Ordered Rate Constitutive Theories in Eulerian Description

By

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Abstract

This research work presents development of ordered rate constitutive theories in Eulerian description for homogeneous, isotropic, compressible and incompressible matter experiencing finite deformation using contravariant, covariant and Jaumann bases. The constitutive theories presented here are applicable to thermoelastic solids, thermofluids and thermoviscoelastic fluids. Due to the inability to monitor material point displacements, and hence strain measures in Eulerian descriptions, the constitutive theories for Cauchy stress tensor utilizing strain measures in Eulerian description are not useful, hence the need for ordered rate constitutive theories presented in this work.

Covariant, contravariant and Jaumann bases identify deformed material lines in the current configuration, thus these bases are possible choices for the development of constitutive theories. Covariant Cauchy stress tensor, contravariant Cauchy stress tensor and Jaumann stress tensor are measures of stress in these bases while Green's strain tensor, Almansi strain tensor and Jaumann strain tensor are conjugate measures of finite strain. Even though strain measures are not defined in Eulerian description, their convected time derivatives in their respective bases are defined. Thus, convected time derivatives of various orders of the Green's strain tensor in covariant basis ($[\gamma_{(k)}]$; $k = 1, 2, \ldots, n$), convected time derivatives of various orders of the Almansi strain tensor in contravariant basis ($[\gamma^{(k)}]$; $k = 1, 2, \ldots, n$) and likewise, Jaumann strain tensor in Jaumann basis ($[{}^{(k)}\gamma^{J}]$; $k = 1, 2, \ldots, n$) are defined and measurable in Eulerian description. These convected time derivatives in their respective bases are symmetric tensors of rank two and are fundamental kinematic tensors, and hence they can be utilized in the derivations of the constitutive theories for the Cauchy stress tensors

in the chosen bases. In addition, we also have convected time derivatives of various orders of the contravariant Cauchy stress tensor $[\bar{\sigma}^{(0)}]$ in contravariant basis ($[\bar{\sigma}^{(k)}]$; $k = 1, 2, \dots, m$), convected time derivatives of various orders of the covariant Cauchy stress tensor $[\bar{\sigma}_{(0)}]$ in covariant basis ($[\bar{\sigma}_{(k)}]$; $k = 1, 2, \dots, m$) and convected time derivatives of various orders of the Jaumann stress tensor $[{}^{(0)}\bar{\sigma}^J]$ in Jaumann basis ($[{}^{(k)}\bar{\sigma}^J]$; $k = 1, 2, \dots, m$. These are also fundamental symmetric tensors of rank two. The ordered rate constitutive theories presented in this work utilize convected time derivatives of upto orders 'n' and 'm' of the strain and stress tensors (i.e. rates) in their respective bases. Thus, there are many possibilities for various rate theories depending upon the choices of the dependent variables in the constitutive theories and their argument tensors. Specific choices of these are made to address specific physics.

In this work we consider homogeneous, isotropic, compressible and incompressible matter with finite deformation, that is in thermodynamic equilibrium during evolution. Thus, conservation laws and thermodynamic principles provide the basis for deriving mathematical models and constitutive theories. Conservation of mass, balance of momenta and the first law of thermodynamics yielding continuity equation, momentum equations and energy equation hold regardless of the constitution of the matter, hence naturally they provide no mechanism for deriving constitutive theories for the stress tensor and the heat vector. Thus, the second law of thermodynamics (entropy inequality) must form the basis for deriving the constitutive theories for the stress tensor and heat vector. The choices of dependent variables in the constitutive theories are made using entropy inequality. The arguments (or eventually argument tensors) of the dependent variables in the constitutive theories are chosen based on the desired physics in conjunction with entropy inequality. When the convected time derivatives of the strain tensor (in a chosen basis) are argument tensors of the dependent variables in the constitutive theories, entropy inequality requires decomposition of the Cauchy stress tensor into equilibrium stress tensor and deviatoric Cauchy stress tensor. Constitutive theories for the equilibrium stress tensor using entropy inequality result in thermodynamic pressure for compressible matter and mechanical pressure for incompressible matter. The conditions resulting from the entropy inequality require that the work expanded due to the deviatoric Cauchy stress tensor be positive but provide no mechanism for deriving constitutive theories for the deviatoric Cauchy stress tensor. The conditions resulting from the entropy inequality also require the scalar product of the heat vector and temperature gradient to be negative which can be used for example to derive the Fourier heat conduction law.

The work presented here utilizes theory of generators and invariants to derive the ordered rate constitutive theories for the deviatoric Cauchy stress tensor and heat vector for homogeneous, isotropic, compressible and incompressible thermoelastic solids, thermofluids and thermoviscoelastic fluids in contravariant, covariant and Jaumann bases. General derivations of rate constitutive theories are specialized to show that (i) generalized hypo-elastic solids, hypo-elastic solids with variable material coefficients are a subset of the general ordered rate constitutive theories of order n for thermoelastic solids (ii) constitutive theories for Newtonian fluids, generalized Newtonian fluids with variable material coefficients such as power law, Carreau-Yasuda model for viscosity, power law, Sutherland law etc. for temperature dependent material coefficients are a subset of the general ordered rate constitutive theories of order n for thermofluids (iii) Maxwell model, Oldroyd-B model, Giesekus model etc with variable transport properties are a subset of the general ordered rate constitutive theories for thermoviscoelastic fluids of orders (m, n). The conditions resulting from entropy inequality, leading to restrictions on the material coefficients, are presented to ensure that the constitutive theories derived using the theory of generators and invariants ensure thermodynamic equilibrium during the evolution. All theories presented here consider finite deformation as well as thermal effects.

A significant aspect of the general theories presented here and the simplifications used to obtain commonly used constitutive theories is that we have clear understanding of the many assumptions employed in obtaining them, hence the possibilities and opportunities for developing better constitutive theories for more precise behaviors of the deforming matter experiencing finite deformation.

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List of symbols:

Tensors of rank zero

α : Mobility factor (polymers)

α_{tm} : Thermal modulus

${}^{\sigma}\alpha^i$: Coefficients in linear combination of combined generators (stress tensor)

${}^{q}\alpha^i$: Coefficients in linear combination of combined generators (heat vector)

η : Specific entropy (Lagrangian description) / First viscosity (fluids and polymers)

$\bar{\eta}$: Specific entropy (Eulerian description)

θ_0 : Temperature in reference configuration

θ : Temperature (Lagrangian description)

$\bar{\theta}$: Temperature (Eulerian description)

κ : Bulk modulus (solids) / Second viscosity (fluids and polymers)

λ : Relaxation time (polymers)

μ : Shear modulus (solids)

ρ_0 : Density in reference configuration

ρ : Density (Lagrangian description)

$\bar{\rho}$: Density (Eulerian description)

Φ : Helmholtz free energy density (Lagrangian description)

$\bar{\Phi}$: Helmholtz free energy density (Eulerian description)

e : Specific internal energy (Lagrangian description)

$\bar{e}$: Specific internal energy (Eulerian description)

I_J : First invariant of Jacobian of deformation

$I\!I_J$: Second invariant of Jacobian of deformation

$I\!I\!I_J$: Third invariant of Jacobian of deformation

$^{q\sigma}\underset{\sim}{I}^i$: Combined invariants of the argument tensors

k : Thermal conductivity

$\bar{p}$: Pressure (Eulerian description)

$\bar{v}$: Specific volume (Eulerian description)

Tensors of rank one

$\boldsymbol{g}$: Temperature gradient (Lagrangian description)

$\bar{\boldsymbol{g}}$: Temperature gradient (Eulerian description)

$\boldsymbol{q}$: Heat vector (Lagrangian description)

$\bar{\boldsymbol{q}}^{(0)}$: Heat vector in contravariant basis

$\bar{\boldsymbol{q}}_{(0)}$: Heat vector in covariant basis

${}^{(0)}\bar{\boldsymbol{q}}^J$: Heat vector in Jaumann basis

${}^{(0)}\bar{\boldsymbol{q}}$: Heat vector (contra- or co-variant or Jaumann)

$\boldsymbol{v}$: Velocity (Lagrangian description)

$\bar{\boldsymbol{v}}$: Velocity (Eulerian description)

$\{{}^{q}\underset{\sim}{G}^i\}$: Combined generators of the argument tensors (heat vector)

Tensors of rank two

$\boldsymbol{\sigma}^*$: First Piola-Kirchhoff stress tensor

${}_e\boldsymbol{\sigma}^*$: First Piola-Kirchhoff equilibrium stress tensor

${}_d\boldsymbol{\sigma}^*$: First Piola-Kirchhoff deviatoric stress tensor

$\boldsymbol{\sigma}^{[0]}$: Second Piola-Kirchhoff stress tensor based on contravariant Cauchy stress tensor

$\boldsymbol{\sigma}_{[0]}$: Second Piola-Kirchhoff stress tensor based on covariant Cauchy stress tensor

$\bar{\boldsymbol{\sigma}}^{(0)}$: Contravariant Cauchy stress tensor

$\bar{\boldsymbol{\sigma}}_{(0)}$: Covariant Cauchy stress tensor

${}^{(0)}\bar{\boldsymbol{\sigma}}^J$: Jaumann stress tensor

${}^{(0)}\bar{\boldsymbol{\sigma}}$: Cauchy stress tensor (contra- or co-variant or Jaumann)

${}^{(0)}_{e}\bar{\boldsymbol{\sigma}}$: Equilibrium Cauchy stress tensor (contra- or co-variant or Jaumann)

${}^{(0)}_{d}\bar{\boldsymbol{\sigma}}$: Deviatoric Cauchy stress tensor (contra- or co-variant or Jaumann)

$[\varepsilon]$: Green's strain

$[\bar{\varepsilon}]$: Almansi strain

$[D]$: Symmetric part of velocity gradient tensor (Lagrangian description)

$[\bar{D}]$: Symmetric part of velocity gradient tensor (Eulerian description)

$[L]$: Velocity gradient tensor

$[W]$: Anti-symmetric part of velocity gradient tensor - Spin tensor

$[\gamma^{(j)}]$: Convected time derivative of order j of Green's strain tensor

$[\gamma_{(j)}]$: Convected time derivative of order j of Almansi strain tensor

$[{}^{(j)}\gamma^{J}]$: Convected time derivative of order j of Jaumann strain tensor

$[{}^{(j)}\gamma]$: Convected time derivative of order j of strain tensor (Green or Almansi or Jaumann)

$[{}_{d}\bar{\sigma}^{(k)}]$: Convected time derivative of order k of contravariant deviatoric Cauchy stress tensor

$[{}_{d}\bar{\sigma}_{(k)}]$: Convected time derivative of order k of covariant deviatoric Cauchy stress tensor

$[{}^{(k)}_{d}\bar{\sigma}^{J}]$: Convected time derivative of order k of covariant deviatoric Jaumann stress tensor

$[{}^{(k)}_{d}\bar{\sigma}]$: Convected time derivative of order k of Cauchy stress tensor (any basis)

$[{}^{\sigma}\underset{\sim}{G}^{i}]$: Combined generators of the argument tensors (stress tensor)

Non-tensor quantities

$[J]$: Jacobian of deformation

$[\bar{J}]$: Inverse of Jacobian of deformation

$[\dot{J}]$: Material time derivative of Jacobian of deformation

$[\ddot{J}]$: Second material time derivative of Jacobian of deformation

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Chapter 1

Introduction, Literature Review and Scope of Work

1.1 Introduction

When mathematical models of deforming matter are derived in Eulerian description, the material particle displacement and hence strain measures are not obtainable. Thus, the constitutive theories for the Cauchy stress tensor in Eulerian description, though they can be derived using strain measures, are not usable as the strains are not defined in Eulerian description. Hence, for the development of constitutive theories in Eulerian descriptions, we must use measures of deformation other than strain measures that are defined in Eulerian description [1–4]. In the derivations of the constitutive theories, regardless of the type of description (i.e. Lagrangian or Eulerian), we must consider deformed material lines in the current configuration. Thus, covariant basis, contravariant basis and Jaumann basis [1] are possible choices for the development of the constitutive theories. For thermoelastic solids, thermofluids and thermoviscoelastic fluids considered in the present work, the development of the constitute theories reduces to constitutive equations for the stress tensor and heat vector.

Green's strain tensor, Almansi strain tensor and Jaumann strain tensor are obvious choices of strain measures in covariant, contravariant and Jaumann bases. Even though these strain measures are not defined in Eulerian description, their convected time derivatives are measurable in their respective bases. We consider $[\gamma^{(k)}]$; $k = 1, 2, \dots, n$, $[\gamma_{(k)}]$; $k = 1, 2, \dots, n$ and $[{}^{(k)}\gamma^J]$; $k = 1, 2, \dots, n$ as the convected time derivatives of upto orders 'n' of the Almansi's strain tensor, Green strain tensor and Jaumann strain tensor in contravariant, covariant and Jaumann bases. These are fundamental kinematic symmetric tensors. Likewise, we also have convected time derivatives of contravariant Cauchy stress tensor $[\bar{\sigma}^{(0)}]$, covariant Cauchy stress tensor $[\bar{\sigma}_{(0)}]$ and Jaumann stress tensor $[{}^{(0)}\bar{\sigma}^J]$ in their respective bases. These are also fundamental symmetric tensors of rank two. The stress measures ($[\bar{\sigma}^{(0)}]$, $[\bar{\sigma}_{(0)}]$ and $[{}^{(0)}\bar{\sigma}^J]$), their convected time derivatives ($[\bar{\sigma}^{(k)}]$; $k = 1, 2, \dots, m$, $[\bar{\sigma}_{(k)}]$; $k = 1, 2, \dots, m$ and $[{}^{(k)}\bar{\sigma}^J]$; $k = 1, 2, \dots, m$) and the convected time derivatives of the strain measures ($[\gamma^{(k)}]$; $k = 1, 2, \dots, n$, $[\gamma_{(k)}]$; $k = 1, 2, \dots, n$ and $[{}^{(k)}\gamma^J]$; $k = 1, 2, \dots, n$) in contravariant basis, covariant basis and Jaumann basis are used to derive the rate constitutive theories for thermoelastic solids, thermofluids and thermoviscoelastic fluids in Eulerian description.

We consider homogeneous, isotropic compressible and incompressible matter undergoing finite deformation. For thermodynamic equilibrium processes considered here, conservation laws and thermodynamic principles must be satisfied. Since conservation of mass, balance of momenta and the first law of thermodynamics must hold regardless of the constitution of the matter, these assume existence of the stress field and heat vector but provide no mechanism for deriving constitutive theories for them. Thus, the second law of thermodynamics i.e. entropy inequality, must form the basis for deriving the constitutive theories. The entropy inequality is used to determine the dependent variables in the constitutive theories. The arguments of the dependent variables are determined based on the desired physics of the deforming matter. It is shown that when the convected time derivatives of the strain tensors (in their respective bases) are argument tensors, entropy inequality requires decomposition of the Cauchy stress tensor into the equilibrium stress tensor and deviatoric Cauchy stress tensor. The constitutive theory for the equilibrium stress tensor is derived using the conditions resulting from entropy inequality yielding equilibrium stress to be thermodynamic pressure for compressible matter and mechanical pressure for incompressible matter. Additionally, the conditions resulting from the entropy inequality require the work expanded due to the deviatoric Cauchy stress tensor to be positive and the scalar product of the heat vector and the temperature gradient to be negative. Using the inequality for the heat vector and the temperature gradient we can derive the constitutive theory for the heat vector. This results in the well known Fourier heat conduction law. However, constitutive theories for the deviatoric Cauchy stress tensor can not be derived using the conditions of positive work expanded.

We utilize the theory of generators and invariants [1, 3–21] to derive ordered rate constitutive theories for the deviatoric Cauchy stress tensor in contravariant, covariant and Jaumann bases. In this approach we must clearly identify the dependent variables in the constitutive theories and their argument tensors. Details of this approach are given in reference [1], but a condensed summary is presented in chapter 2. In the following, we first present a brief literature review of the pertinent published work. This is followed by the scope of the research work.

1.2 Literature Review

The concepts of convected time derivatives, the objective rates of the stress and strain tensors and their use in the constitutive models can be traced back to Jaumann (1905), Oldroyd B (1958), Giesekus (1962) and others in references [22–28] in connection with solid matter as well as polymeric liquids. Hypo-elastic constitute laws [29–34] are a special form of the rate constitutive equations for elastic solid matter. Upper convected, lower convected, Jaumann rate constitutive equations have been used commonly in a large volume of published work on mathematical models [3, 4, 22, 29, 30, 35] and numerical computations [36, 37] for solid matter. In many of these works the emphasis is on the use of the constitute models as opposed to their origin and the details of their derivations.

Broadly speaking, all materials can be classified into two categories: materials without memory and those with memory. A material without memory has no recollection of how the current configuration is arrived at. For such materials the only correspondence of the current configuration is to the reference configuration. On the other hand, materials with memory have recollection of past events. If the recollection is limited to immediately preceding few events, then we say that the material has fading memory. Materials with memory naturally exhibit relaxation phenomenon. Such materials upon cessation of the disturbance require finite amount of time to resume relaxed (unstressed) state dependent on the characteristic constant of the material. Incompressible and compressible Newtonian fluids, generalized Newtonian fluids and many other fluids described by higher order rate theories using convected time derivatives of the strain tensors are examples of fluids without memory. Thermoviscoelastic fluids are fluids with memory.

Newton's law of viscosity for incompressible and compressible fluids are well known and widely used as constitutive equations for incompressible and compressible thermoviscous fluids (Newtonian fluids) [38, 39]. The constitutive models for generalized Newtonian fluids, such as power law and Carreau-Yasuda model, are extensions of the constitutive models for Newtonian fluids in which the medium viscosity is assumed to depend on the deformation field [22]. The developments in continuum mechanics in the last three decades have been overwhelming [2, 40–42]. These have been largely initiated and focused on solid matter with applications to liquids and gases. While the basic definitions and the measures such as kinematics of deformation, measures of stresses, strains, their rates etc. do not distinguish between the specific nature of the matter and hence are equally applicable to solids, liquids and gases, this is not the case in the development of the constitutive theory due to the obvious fact that the constitutive equations are mathematical descriptions that are specific for a given type of matter. Thus the developments in the constitutive theory for solid matter can be useful when considering liquids and gases but these can not be imported in their entirety and used for liquids and gases. This is primarily due to the fact that the composition and behavior of liquids and gases are drastically different than solids, hence completely new considerations may be necessary in the development of the constitutive theory for such matter compared to those for solid matter.

The thermoviscoelastic fluids or polymeric fluids are both viscous and elastic. Such fluids consist of a solvent and a polymer. The solvent is a very dilute solution that may be primarily viewed as Newtonian fluid. Its composition is due to short chain molecules. The polymer on the other hand consists of long chain molecules. It has its own viscosity in addition to elasticity. In thermoviscoelastic fluids, the elastic effects are primarily due to the polymer. When a polymeric fluid is subjected to a disturbance, the motion of the polymer molecules is complex (Brownian motion [43, 44]). The polymeric fluids can be classified in two broad categories: dilute polymeric fluids and dense polymeric fluids or polymer melts. Compressibility in polymeric fluids is only important at very high pressures. Generally, polymeric fluids are treated as incompressible, hence it is appropriate to say polymeric liquids. Dilute polymeric fluids are primarily much like Newtonian fluids but with some elastic effects, i.e., the behavior is dominated by viscous effects. In such fluids the solvent viscosity is dominant, i.e., much higher than the polymer viscosity. Polymer melts on the other hand are dense polymeric fluids whose behavior is dominated by elastic effects. In such fluids the polymer viscosity is much higher than the solvent viscosity. Polymeric fluids are of significant industrial importance.

The first attempt to obtain constitutive equations for viscoelastic liquids appears to have been due to Maxwell [45]. Later these were generalized to remove the small displacement assumption [22]. Maxwell constitutive model is a ‘linear viscoelastic model’. Using Maxwell model as a basis, the Jeffreys model is obtained by adding time derivative of the symmetric part of the velocity gradient tensor [22, 46]. Generalization of the Maxwell model is obtained by superposition of a series of Maxwell models [22]. It is commonly accepted [22] that linear viscoelastic models have many limitations: (1) They can not describe shear rate dependent viscosity (2) They can not describe normal stress behavior (3) They fail to describe small-strain phenomena if it is accompanied by large displacements due to rigid rotations. These lead to the development of 'quasi-linear differential models'. The Oldroyd-B [47] model falls into this category. Deficiencies of these models in describing realistic physical flow phenomena in polymer melts lead to the development of 'non-linear differential constitutive models' for polymeric fluids. Giesekus model [48] and PTT model [49, 50] fall into this category. Many other constitutive models have been proposed for polymeric fluids (see reference [22]). The fundamental driving principles behind these models have been anisotropic drag due to Brownian motion of polymer molecules and their networks and the kinetic theory [22, 51].

Polymeric fluids at a macro scale are viewed as isotropic homogeneous continuous media. Thus, in our view, the constitutive theory for such fluids must be derivable using principles and axioms of continuum mechanics. In fact, Maxwell constitutive model has been derived by Eringen in references [3, 4] using the theory of generators and invariants. Since the thermoviscoelastic fluids have memory and hence exhibit relaxation phenomenon, it can be shown [1, 22] that the very least we must consider in the constitutive theory is the first convected time derivative of the chosen strain tensor and the stress tensor itself as arguments of the first convected time derivative of the stress tensor.

The first account of the phrase 'ordered material' seems to have been first introduced by Bird [22] in connection with 'retarded motion expansion' defined as a deviation from Newtonian fluids. It was advocated that retarded motion expansion is the correct constitutive equation for flows in which rate-of-strain tensor and its time derivatives are small. The works in reference [25–28] are the basis for the presentations of ordered fluids by Bird [22]. In these works, the stress tensor is considered as a polynomial in the convected time derivatives of progressively increasing orders of the strain rate tensors.

The ordered rate constitutive theories presented here are a generalization of the concept presented by Eringen in references [3,4] using the theory of generators and invariants. To our knowledge, derivations of other constitutive models in Eulerian description based on principles of continuum mechanics are not reported in the published works. While in simplified cases, the polynomial approach [22] may yield the same results as presented in the current work, in general, we observe three fundamental problems in this approach: (i) Lack of derivation based on the second law of thermodynamics which necessitates decomposition of the Cauchy stress tensor into equilibrium stress and deviatoric stress. While the equilibrium stress is deterministic from the Clausius-Duhem inequality, the deviatoric stress is not. For the deviatoric stress one must use the theory of generators and invariants as opposed to the polynomial approach [22,25–28]. (ii) The constitutive theories must also address the constitutive equation for the heat vector. (iii) Co- and contra-variant bases as well as Jaumann basis and the development of the constitutive theories in these bases and the differences in the resulting constitutive equations must be addressed. It is instructive to examine the derivations of the currently used models based on continuum mechanics axioms and principles as a subset of ordered rate constitutive theories as it may suggest new possibilities for improved constitutive models.

In references [52–54] various aspects of conjugate stress and strain measures, constitutive inequalities and stability expressed in terms of stress and strain rates are considered in the development of the constitutive theories. The postulates of material behavior stating lower bounds to the work expanded are reported to be in agreement with observed behaviors but are not supported by thermodynamic considerations (as stated in reference [52]). The theories presented in the current work consider condition(s) resulting from the second law of thermodynamics which only requires that the work expanded must be positive.

1.3 Scope of Work

The present work focuses on the development of general rate constitutive theories for ordered thermoelastic solids, ordered thermofluids and ordered thermoviscoelastic matter in Eulerian description. The research considers both compressible and incompressible in co- and contra-variant bases as well as using Jaumann rates based on the principles and axioms of continuum mechanics. At the onset of the development of the constitutive theory, the choice of stress tensor and heat vector as dependent variables is rather obvious. We begin all developments with the second law of thermodynamics (entropy inequality), an essential thermodynamic law for the development of the constitutive theory if the deforming matter is to be in thermodynamic equilibrium. The Cauchy stress tensor is decomposed into equilibrium stress and deviatoric stress as necessitated by the entropy inequality. The constitute equation for the equilibrium stress for both compressible and incompressible cases is established using entropy inequality. The constitutive equations for the deviatoric Cauchy stress can not be determined using entropy inequality as it provides no mechanism for doing so except that it requires the work expanded due to the deviatoric Cauchy stress to be positive.

In case of ordered thermoelastic solids, we use the theory of generators and invariants to (i) establish a most general form of rate constitutive theory in which the first convected time derivative of the deviatoric Cauchy stress tensor and the heat vector can be a functions of the convected time derivatives of up to order 'n' of the conjugate strain tensor, density, temperature and temperature gradient, (ii) specialize the general theory presented in (i) to second order thermoelastic solids, and (iii) further specialize the theory presented in (ii) to first order thermoelastic solids and demonstrate that the general constitutive theory of ordered thermoelastic solids of order one reduces to the well known hypo-elasticity with further assumptions. All derivations and details for (i) - (iii) are presented using contravariant and covariant bases as well as Jaumann rates for incompressible and compressible thermoelastic solids. Discussion and arguments are presented for validity and usefulness of contravariant, covariant and Jaumann rate constitutive equations.

In the case of ordered thermofluids, we also use the theory of generators and invariants to establish a general rate constitutive theory in which the deviatoric Cauchy stress tensor and the heat vector can be a functions of the convected time derivatives of up to order 'n' of the conjugate strain tensor, density, temperature and temperature gradient. The general theory is simplified to obtain the constitutive equations for the well known generalized Newtonian and Newtonian fluids. The developments of the constitutive theory are presented using co- and contra-variant bases as well as Jaumann stress and strain rates for incompressible and compressible cases. The consequences of the choice of basis are discussed and illustrated in the general derivation as well as specialized cases.

The Maxwell, Giesekus and Oldroyd-B constitutive models used currently are derived as special cases of the general rate theory for ordered thermoviscoelastic fluids. The theory of generators and invariants provides the foundation for establishing the constitutive theories for the deviatoric Cauchy stress tensor heat vector. In a chosen basis, the convected time derivative of order 'm' of the deviatoric Cauchy stress tensor is expressed in terms of the density, temperature, temperature gradient, convected time derivatives of the conjugate strain tensor (Almansi or Green or Jaumann) of up to order 'n' and the convected time derivatives of the deviatoric Cauchy stress tensor of up to order '$m-1$' as argument tensors. In case of the heat vector we also consider density, temperature, temperature gradient, convected time derivatives of the strain tensor of up to order 'n' and convected time derivatives of the deviatoric Cauchy stress tensor of up to order '$m-1$' as argument tensors. The general derivation of the rate constitutive theory for the deviatoric Cauchy stress and the heat vector are specialized to derive upper convected (contravariant basis), lower convected (covariant basis) and Jaumann rate constitutive equations commonly used for for Maxwell, Giesekus and Oldroyd-B fluids. The derivation presented in this work shows that the deviatoric Cauchy stress tensor (in either contra- or co-variant basis and using Jaumann rates) naturally results as a dependent variable in the constitutive theory as opposed to deviatoric Cauchy stress tensor based on polymer stress as used currently in the Giesekus constitutive model. Numerical studies are presented for a dense polymeric liquid using the Giesekus constitutive model derived in the present work as well as currently used Giesekus constitutive model.

Chapter 2

Ordered Rate Constitutive Theories in Eulerian Description

2.1 Introduction

The motivation for the development of the ordered rate constitutive theories in Eulerian description is multifold. First, we note that based on the constitutive theories for thermoelastic solids in Eulerian description, these theories are not usable due to the fact that in Eulerian descriptions the material point displacements are not measurable, hence the strain measures used in these theories are not defined. Thus, for the development of the constitutive theories for all deforming matter (solid, liquid or gas) in Eulerian description we must take a more fundamental approach. In the developments considered in this chapter and chapters 3 to 5, we only consider homogeneous and isotropic matter. We list some important considerations in the following [1, 55–57]:

(i) For *thermodynamic equilibrium* during evolution of the deforming matter, entropy inequality must form the basis for deriving the constitutive theories.

(ii) In Eulerian description we consider current configuration and a fixed position $\bar{x}_i$ in it. If we consider orthogonal material lines (oA, oB, oC) in the reference configuration at a point o (with coordinates x_i), then upon deformation the material lines in the current configuration become curvilinear at position $\bar{o}$ (with coordinates $\bar{x}_i$) shown in figure 2.1. The tangent vectors to the deformed material lines $\bar{o}\tilde{x}_i$ are called covariant base vectors $\bar{o}\bar{A}$, $\bar{o}\bar{B}$, $\bar{o}\bar{C}$.

(iii) If we consider an elementary tetrahedron in the reference configuration whose faces are formed by the orthogonal material lines oA, oB and oC, then upon deformation, this tetrahedron deforms into $\bar{o}\bar{A}\bar{B}\bar{C}$ whose faces are formed by the covariant base vectors.

(iv) The covariant basis defines tangent vectors to the deformed material lines in the current configuration at a material point. We could also consider a basis formed by the vectors orthogonal to the faces of the deformed tetrahedron that are formed by the covariant base vectors. This basis is called contravariant basis. As shown in earlier chapter contravariant basis is reciprocal to the covariant basis and hence identifies deformed material lines as well.

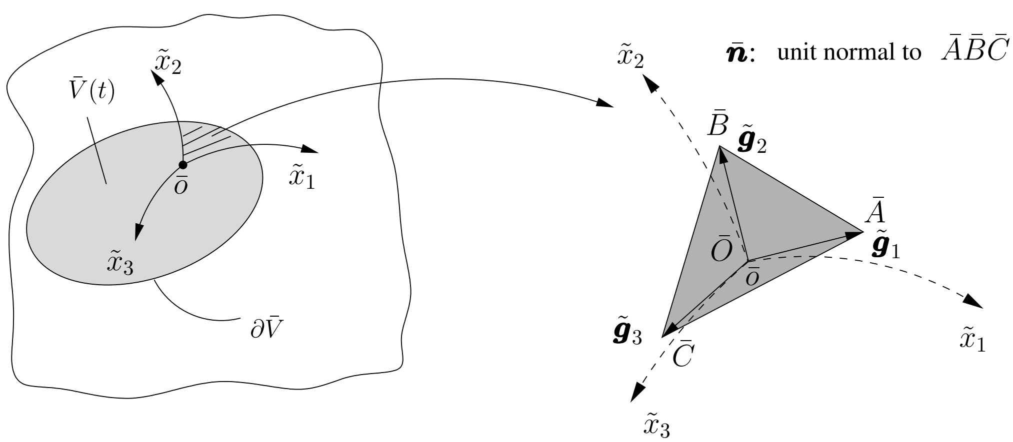
(v) Thus, covariant and contravariant bases are natural choices for the development of the constitutive theories in the Eulerian description due to the fact these bases identity deformed material lines in the current configuration.

(vi) Next, we must identify the measures of stresses and finite strain in these two bases that are *conjugate in the sense of energy*. The contravariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ and covariant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}_{(0)}$ are the obvious choices for the measures of stresses in contravariant and covariant bases. Almansi strain tensor $[\bar{\varepsilon}]$ and Green's strain tensor $[\varepsilon]$ are the conjugate measures of finite strain in contravariant and covariant bases.

(vii) As mentioned earlier, the strain measures $[\bar{\varepsilon}]$ and $[\varepsilon]$ are not defined in the Eulerian description, hence we must consider convected time derivatives of strain tensors $[\bar{\varepsilon}]$ and $[\varepsilon]$ in contra- and co-variant bases in the development of the constitutive theories as these are defined in the Eulerian description. Likewise, we can also consider the convected time derivatives of the stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\bar{\boldsymbol{\sigma}}_{(0)}$ in contra- and co-variant bases. The resulting constitutive theories utilizing these are *rate constitutive theories* as these employ rates of stress and strain tensors. The use of the term *ordered* will be clear when we discuss specific details of the development of these theories.

(viii) The choices of specific stress and strain convected time derivatives in various theories is of course dependent on the desired physics and are discussed in a subsequent section as well as in chapters 3 to 5.

(ix) We reiterate that the rate constitutive theories must satisfy all axioms or principles of constitutive theory. The *axiom of equipresence*, *axiom of admissibility*, *axiom of objectivity (frame and form invariance)* and the *axiom of smooth neighborhood* being some of the most important once that are used directly in the development of the rate constitutive theories.



(a) Current configuration

(b) Deformed tetrahedron with co-variant base vectors

Figure 2.1: Deformed elementary tetrahedron in the current configuration

2.2 Preliminary considerations in the rate constitutive theories [1,55–57]

Since conservation of mass, balance of momenta and conservation of energy are independent of the constitution of the deforming matter, we must consider the fourth conservation law, the second law of thermodynamics in the development of the rate constitutive theories. Even though we are only concerned with rate constitutive theories in Eulerian description, it is instructive to consider entropy inequality in Lagrangian as well as Eulerian description. Some of the deductions are more easily seen in Lagrangian description due to the simplicity of the material derivatives. Recall entropy inequalities in Eulerian and Lagrangian descriptions.

$$\bar{\rho}\Big(\frac{D\bar{\Phi}}{Dt} + \bar{\eta}\frac{D\bar{\theta}}{Dt}\Big) + \frac{\bar{q}_i^{(0)}\bar{g}_i}{\bar{\theta}} - \bar{\sigma}_{ij}^{(0)}\frac{\partial \bar{v}_i}{\partial \bar{x}_j} \leq 0 \qquad \text{Eulerian description} \tag{2.1}$$

$$\rho_0\Big(\frac{\partial \Phi}{\partial t} + \eta\frac{\partial \theta}{\partial t}\Big) + \frac{|J|\underset{\sim}{q}_i\underset{\sim}{g}_i}{\theta} - \sigma_{ki}^{*}\dot{J}_{ik} \leq 0 \qquad \text{Lagrangian description} \tag{2.2}$$

in which all quantities have their usual meaning and $\dot{J}_{ik}$ is the material derivative of $[J]$. (2.1) has been expressed using contravariant Cauchy stress tensor. We could have also used $\bar{\boldsymbol{\sigma}}_{(0)}$, covariant Cauchy stress tensor, or ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$, Jaumann stress tensor. $\boldsymbol{\sigma}^*$ in (2.2) is first Piola-Kirchhoff stress tensor that can be easily expressed in terms of second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ or $\boldsymbol{\sigma}_{[0]}$. In the following discussion it is immaterial whether we consider (2.1) or (2.2) due to the fact all measures used in (2.1) are related to those used in (2.2) through $[J]$ or $[\bar{J}]$.

2.2.1 Choice of independent variables in the constitutive theories

Based on the *axiom of casualty* we consider motion of the material points of a body and their temperatures as self-evident observable effect in every thermomechanical behavior of matter, hence these are the *independent variables in the development of the constitutive theory*. The remaining quantities, other than those that can be derived using motion and temperature of material points that enter the expression of entropy generation or production are the causes or *dependent variables in the development of the constitutive theory*.

As an example, in case of thermomechanical behavior of deforming matter, based on this principle, the *constitutive independent variables* are

$$\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\boldsymbol{x}, t) \quad , \quad \boldsymbol{x} = \boldsymbol{x}(\bar{\boldsymbol{x}}, t) \qquad ; \qquad \bar{\theta} = \bar{\theta}(\bar{\boldsymbol{x}}, t) \quad , \quad \theta = \theta(\boldsymbol{x}, t) \tag{2.3}$$

Now the velocity can be derived using time derivatives of $\bar{\boldsymbol{x}}$, density in the current configuration is deterministic from the conservation of mass or continuity equation. Thus, in describing the entropy production, the quantities that remain should be considered as *constitute dependent variables* and must be expressed in terms of: $\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\boldsymbol{x}, t), \boldsymbol{x} = \boldsymbol{x}(\bar{\boldsymbol{x}}, t)$; $\bar{\theta} = \bar{\theta}(\bar{\boldsymbol{x}}, t), \theta = \theta(\boldsymbol{x}, t)$.

2.2.2 Choice of dependent variables in the constitutive theories

From entropy inequality we note that it contains stress $\bar{\boldsymbol{\sigma}}$ (contra- or co-variant Cauchy stress tensor), heat vector $\bar{\boldsymbol{q}}$ (contra- or co-variant), specific Helmholtz free energy density $\bar{\Phi}$ and specific entropy $\bar{\eta}$ in Eulerian description, and $\boldsymbol{\sigma}$ (first Piola-Kirchhoff stress, or second Piola-Kirchhoff stress tensor), $\boldsymbol{q}$, Φ and η in Lagrangian description. Choice of specific internal energy $\bar{e}$, $\bar{\eta}$ or $\bar{\Phi}$, $\bar{\eta}$ in Eulerian description is a matter of preference as they are related through $\bar{\Phi}$. Likewise choice of e, η or Φ, η in Lagrangian description is a matter of preference as well, as these are also related through Φ. We consider $\bar{\Phi}$, $\bar{\eta}$ and Φ, η in the details that follow. Regardless of which choice is made, i.e., e, η or Φ, η, the constitutive theory is unaffected as Φ and e are related. In the work presented here we only consider *homogeneous and isotropic matter* in which a material point represents each material point in the entire volume of the matter. Thus, for homogeneous and isotropic matter, the constitutive theory derived at a material point is valid for the entire volume of matter.

For simple materials such as elastic solids, thermoelastic solids, thermoviscoelastic solids, Newtonian fluids, polymers etc. the objective of constitutive theories is to provide a mathematical foundation for quantitatively establishing the stress field and heat vector in the deforming matter as functions of tensors that are measures of the physics of the deforming matter in the current configuration. Thus the stress tensor and heat vector are undoubtedly the dependent variables in the constitutive theory. Based on the comment made in the previous paragraph, we also choose Φ and η (or $\bar{\Phi}$ and $\bar{\eta}$) as additional dependent variables in the constitutive theories i.e. in addition to the stress tensor and heat vector.

Thus the choice of dependent variables in the constitutive theories is $\boldsymbol{\sigma}$, $\boldsymbol{q}$, Φ, η (or $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{q}}$, $\bar{\Phi}$, $\bar{\eta}$). The dependent variables $\boldsymbol{\sigma}$, $\boldsymbol{q}$, Φ, η or ($\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{q}}$, $\bar{\Phi}$, $\bar{\eta}$) are functions of arguments describing the physics of the deforming matter. At the onset we consider the principle or axiom of equipresence and take into account all possible measures of deformation as arguments of $\boldsymbol{\sigma}$, $\boldsymbol{q}$, Φ, η. Some of these may be ruled out at a later stage due to other considerations.

2.2.3 Choice of arguments for the dependent variables in the constitutive theories

For the specific matter under consideration, the choice of arguments for the dependent variables $\boldsymbol{\sigma}$, $\boldsymbol{q}$, Φ and η (or $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{q}}$, $\bar{\Phi}$ and $\bar{\eta}$) in the constitutive theories consistent with the desired physics and in agreement with the axioms of the constitutive theory, is obviously the most crucial and the most important aspect of the development of the constitutive theory. For the rate constitutive theories considered in this chapter, the details of the choices of these arguments are described here as well as in the subsequent sections and chapters.

For simplicity, consider Lagrangian description i.e. (2.2). Based on the *axiom of equipresence* we consider all possible measures of deformation as arguments of $\boldsymbol{\sigma}$, $\boldsymbol{q}$, Φ and η. The Jacobian of deformation $[J]$ is fundamental in the kinematics of deformation and hence must be an argument in each of the four dependent variables. Since we need to consider convected time derivatives of the strain tensors in the rate theories, $[\dot{J}]$ (time or material derivative of $[J]$) must also be an argument of the dependent variables in the constitutive theory. We are also considering thermal effects, hence temperature θ is obviously an argument. In addition to these three, we also consider $\boldsymbol{g}$, the temperature gradient as an argument. Thus at the onset we have

$$
\begin{aligned}
\boldsymbol{\sigma} &= \boldsymbol{\sigma}([J],[\dot{J}],\theta,\boldsymbol{g}) \\
\boldsymbol{q} &= \boldsymbol{q}([J],[\dot{J}],\theta,\boldsymbol{g}) \\
\Phi &= \Phi([J],[\dot{J}],\theta,\boldsymbol{g}) \\
\eta &= \eta([J],[\dot{J}],\theta,\boldsymbol{g})
\end{aligned}
\tag{2.4}
$$

If in (2.4) the independent variables are (x_i, t), then (2.4) are Lagrangian or material description in which case $\boldsymbol{\sigma}$ may represent first Piola-Kirchhoff stress $\boldsymbol{\sigma}^*$, or second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ or $\boldsymbol{\sigma}_{[0]}$. On the other hand, if the independent variables are $(\bar{x}_i, t)$, then these are Eulerian descriptions in which case $\boldsymbol{\sigma}$ may represent contra- or co-variant Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$ or $\bar{\boldsymbol{\sigma}}_{(0)}$ or Jaumann stress tensor ${}^{(0)}\bar{\boldsymbol{\sigma}}^J$.

2.2.4 Possible approaches to the development of constitutive theories [1]

For a specific matter under consideration, once the arguments for $\boldsymbol{\sigma}$, $\boldsymbol{q}$, Φ, η that are in agreement with the axioms of the constitutive theories and physics of deforming matter are established, we can consider the following two possible approaches depending upon the type of matter and the type of description, i.e., Lagrangian or Eulerian.

Approach (1)

Due to the *axiom of admissibility*, all constitutive theories must satisfy conservation laws. Conservation of mass, balance of momenta and conservation of energy are independent of the constitution of the matter. Their derivation assumes existence of the stress field and heat vector. Thus what remains is the second law of thermodynamics or Clausius-Duhem inequality. That is, all constitutive theories must satisfy entropy inequality. Said differently, if we use entropy inequality to derive constitutive theories then they will naturally satisfy the second law of thermodynamics. In continuum mechanics this is the fundamental approach for deriving constitutive theories for thermoelastic solids. Using this approach it is possible:

(a) To derive simple constitutive theories for the heat vector such as Fourier heat conduction law.

(b) For elastic and thermoelastic solid matter, the stress field in a deforming matter can be established in Lagrangian and Eulerian descriptions in terms of chosen strain measures.

(c) When the mathematical models are derived for thermoelastic solid matter using Eulerian descriptions, the entropy inequality also provides mechanism to derive constitutive theories for stress tensor, but such constitutive theories are not usable due to the fact that in Eulerian descriptions material point displacements are not known, hence the strain tensors used in these constitutive theories are not measurable. Thus, the need for rate constitutive theories in Eulerian description. At this stage it is not clear whether the conditions resulting from the entropy inequality are sufficient for development of rate constitutive theories.

(d) This approach of deriving constitutive theories strictly using conditions resulting from the entropy inequality obviously has *thermodynamic basis* as the constitutive theories in this case are derived using the conditions resulting from the second law of thermodynamics without violating the other conservation laws.

Approach (2)

In deforming matter in which entropy inequality does not provide an explicit mechanism for deriving constitutive theories, we use an alternate approach. By examining the constitutive equations for elastic and thermoelastic solid matter in Lagrangian description derived using entropy inequality [1, 55], we note that the expressions for the stress tensor are a linear combination of the *combined generators* [3–21] of the argument tensors. This observation suggests an approach for deriving constitutive theories in which the stress tensor is expressed as a linear combination of its combined generators of the argument tensors. The coefficients in the linear combination are functions of the *combined invariants* of the argument tensors (and others), and are determined using their Taylor series expansions about an immediately preceding known configuration (as in approach (1)). Perhaps a simpler way to explain the same idea is to say that the stress tensor and heat vector are in some spaces. These spaces have bases which can be established using the argument tensors of the stress tensor and the heat vector. Once we know the bases, we can express the stress tensor and heat vector as a linear combination of the respective basis. The combined generators of the argument tensors of the stress tensor and the heat vector indeed are the bases of the spaces in which these can be defined by using their linear combinations. A basis i.e., the collection of combined generators of the argument tensors for the stress tensor or heat vector, that contains the smallest possible number of members is called *minimal basis* or *integrity* (section 2.8).

This approach of deriving constitutive theories uses principles and concepts of continuum mechanics and hence has *continuum mechanics foundation* but may lack thermodynamic basis if the conditions resulting from the entropy inequality are not satisfied.

***Remarks*:**

The two approaches listed above ((1) and (2)) provide a unified framework for the development of the constitutive theories. We do remark that:

1 Approach (1) is strictly in accordance with entropy inequality and hence has thermodynamic basis.

2 Approach (2) has continuum mechanics foundation in the sense that it utilizes continuum mechanics concepts and axioms of the constitutive theories but can be viewed to lack thermodynamic basis due to the fact that the constitutive theories in this case are not derived directly using the second law of thermodynamics or using the conditions resulting from it.

3 In approach (2) we do have to ensure that the conditions resulting from the entropy inequality are not violated.

4 The decision on whether approach (1) will provide a mechanism for rate constitutive theories or whether we need to consider approach (2) can only be made by considering entropy inequality, keeping in mind relations (2.4).

2.3 Entropy inequality: Lagrangian description [1,3,4,55–57]

Whether we consider entropy inequality in Lagrangian or Eulerian description is immaterial due to the fact that all measures and their descriptions are transformable from one description to the other using $[J]$ and $[\bar{J}]$. For the sake of simplicity and clarity we consider entropy inequality in Lagrangian description (2.2). Using $\Phi(\cdot)$ with its arguments in (2.4). We can write

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial J_{ik}} \dot{J}_{ik} + \frac{\partial \Phi}{\partial \dot{J}_{ik}} \ddot{J}_{ik} + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \frac{\partial \Phi}{\partial g_i} \dot{g}_i \tag{2.5}$$

Substituting (2.5) in (2.2)

$$\rho_0 \Big(\frac{\partial \Phi}{\partial J_{ik}} \dot{J}_{ik} + \frac{\partial \Phi}{\partial \dot{J}_{ik}} \ddot{J}_{ik} + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \eta \frac{\partial \theta}{\partial t} \Big) + \frac{|J| \underset{\sim}{q_i} \underset{\sim}{g_i}}{\theta} - \sigma^*_{ki} \dot{J}_{ik} \leq 0 \tag{2.6}$$

or $$\rho_0 \frac{\partial \Phi}{\partial \dot{J}_{ik}} \ddot{J}_{ik} + \Big(\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - \sigma^*_{ki} \Big) \dot{J}_{ik} + \rho_0 \Big(\frac{\partial \Phi}{\partial \theta} + \eta \Big) \dot{\theta} + \frac{|J| \underset{\sim}{q_i} \underset{\sim}{g_i}}{\theta} + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i \leq 0 \tag{2.7}$$

In order for (2.7) to hold for arbitrary (but admissible) $[\ddot{J}]$, $\dot{\boldsymbol{g}}$, $\dot{\theta}$, the following must hold:

$$\rho_0 \frac{\partial \Phi}{\partial \dot{J}_{ik}} = 0 \qquad \Rightarrow \qquad \frac{\partial \Phi}{\partial \dot{J}_{ik}} = 0 \tag{2.8}$$

$$\rho_0 \frac{\partial \Phi}{\partial g_i} = 0 \qquad \Rightarrow \qquad \frac{\partial \Phi}{\partial g_i} = 0 \tag{2.9}$$

$$\rho_0 \Big(\frac{\partial \Phi}{\partial \theta} + \eta \Big) = 0 \qquad \Rightarrow \qquad \frac{\partial \Phi}{\partial \theta} + \eta = 0 \tag{2.10}$$

and $$\Big(\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - \sigma^*_{ki} \Big) \dot{J}_{ik} + \frac{|J| \underset{\sim}{q_i} \underset{\sim}{g_i}}{\theta} \leq 0 \tag{2.11}$$

(2.8) - (2.11) are fundamental relations from second law of thermodynamics (entropy inequality).

***Remarks*:**

(1) Equation (2.8) implies that Φ is not a function of $[\dot{J}]$.

(2) Equation (2.9) implies that Φ is not a function of $\boldsymbol{g}$ either.

(3) Based on (2.10), η is not a dependent variable in the constitutive theories as $\eta = -\frac{\partial \Phi}{\partial \theta}$, hence η is deterministic from Φ.

(4) The inequality in the last equation (2.11) is essential in the form it is stated. For example

$$\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - \sigma^*_{ki} = 0 \qquad \text{and} \qquad \frac{|J| \underset{\sim}{q}_i \underset{\sim}{g}_i}{\theta} \leq 0$$

are inappropriate due to the fact that these imply that $[\sigma^*]$ is not a function of $[\dot{J}]$ (as Φ is not a function of $[\dot{J}]$) which is contrary to (2.4). We note that (2.11) in its stated form is unable to provide us further details regarding the derivation of the constitutive theory for $[\sigma^*]$ and $\boldsymbol{q}$.

Stress decomposition

In order to alleviate the situation discussed in remark (4), we consider decomposition of $[\sigma^*]$ into equilibrium stress $[{}_e\sigma^*]$ and deviatoric stress $[{}_d\sigma^*]$, i.e.

$$[\sigma^*] = [{}_e\sigma^*] + [{}_d\sigma^*] \tag{2.12}$$

in which we have the following:

$$[{}_e\sigma^*] = [{}_e\sigma^*([J], [0], \theta, \boldsymbol{g})] \tag{2.13}$$

$$[{}_d\sigma^*] = [{}_d\sigma^*([J], [\dot{J}], \theta, \boldsymbol{g})] \tag{2.14}$$

$$\text{and} \quad [{}_d\sigma^*] = [{}_d\sigma^*([J], 0, \theta, 0)] = 0 \tag{2.15}$$

That is, $[{}_e\sigma^*]$ is not a function of $[\dot{J}]$ and $[{}_d\sigma^*]$ vanishes when $[\dot{J}]$ and $\boldsymbol{g}$ are zero. Substituting (2.12) into (2.11) gives

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - {}_e\sigma^*_{ki} - {}_d\sigma^*_{ki}\right)\dot{J}_{ik} + \frac{|J|\underset{\sim}{q_i}\underset{\sim}{g_i}}{\theta} \leq 0 \tag{2.16}$$

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - {}_e\sigma^*_{ki}\right)\dot{J}_{ik} - {}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J|\underset{\sim}{q_i}\underset{\sim}{g_i}}{\theta} \leq 0 \tag{2.17}$$

Since Φ is not a function of $[\dot{J}]$ and neither is $[{}_e\sigma^*]$ (due to (2.13)), then $[{}_e\sigma^*]$ must be derivable from

$${}_e\sigma^*_{ki} = \rho_0 \frac{\partial \Phi}{\partial J_{ik}} \qquad \text{or} \qquad [{}_e\sigma^*]^T = \rho_0 \frac{\partial \Phi}{\partial [J]} \tag{2.18}$$

Using (2.18), the inequality (2.17) reduces to

$$-{}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J|\underset{\sim}{q_i}\underset{\sim}{g_i}}{\theta} \leq 0 \tag{2.19}$$

If we assume (as done routinely to derive Fourier heat conduction law)

$$\frac{|J|\underset{\sim}{q_i}\underset{\sim}{g_i}}{\theta} \leq 0 \tag{2.20}$$

then (2.19) is satisfied if the following holds:

$${}_d\sigma^*_{ki}\dot{J}_{ik} > 0 \tag{2.21}$$

Equation (2.21) requires that *the work expanded due to the deviatoric stress tensor must be positive*. Thus (2.12) can be written as

$$\sigma^*_{ij} = \rho_0 \frac{\partial \Phi}{\partial J_{ji}} + {}_d\sigma^*_{ij}([J], [\dot{J}], \theta, \boldsymbol{g}) \tag{2.22}$$

Furthermore, based on (2.8) and (2.9) we can write

$$\Phi = \Phi([J], \theta) \tag{2.23}$$

$$\boldsymbol{q} = \boldsymbol{q}([J], [\overset{\bullet}{J}], \theta, \boldsymbol{g}) \tag{2.24}$$

Thus, we note that when $[\overset{\bullet}{J}]$ is an argument of the dependent variables in the constitutive theory:

(i) The entropy inequality requires stress decomposition into equilibrium and deviatoric stress in order to proceed further with the development of the constitutive theories.

(ii) Based on the conditions resulting from the entropy inequality i.e. (2.22), equilibrium stress is deterministic from the Helmholtz free energy density, but the constitutive theory for the deviatoric stress is not.

Thus, in case of rate constitutive theories, the entropy inequality can only take as far as (2.22) - (2.24). We consider details of the constitutive theory for equilibrium and deviatoric stress tensors in the following. Derivation of the Fourier heat conduction law for the heat vector is straight forward based on (2.20). A more general derivation of the constitutive theory for the heat vector is considered in the subsequent sections. We consider further details of the arguments in (2.22) - (2.24) in the following.

2.4 Further considerations regarding the arguments in equations (2.22) - (2.24) [1,55–57]

2.4.1 Lagrangian description

So far we have considered entropy inequality in Lagrangian description (for convenience) and the conditions resulting from it. In Eulerian description we monitor the state of deforming matter at a fixed location $\bar{x}_i$ in the current configuration. Thus, transformation of the reference frame by a unimodal (orthogonal) matrix can not be detected by the subsequent thermodynamical deformation. Consider Lagrangian description.

If x-frame changes to x'-frame via

$$\{x'\} = [R]\{x\} \tag{2.25}$$

$$\therefore \qquad [J'] = [J][R]^T \tag{2.26}$$

then, based on the principle of frame invariance

$$\Phi([J],\theta) = \Phi([J'],\theta) = \Phi([J][R]^T,\theta) \tag{2.27}$$

must hold and likewise, the principle of frame invariance must also hold for the stress tensor and heat vector. But this is only possible if Helmholtz free energy density Φ, the stress tensor and heat vector depend upon the invariants I_J, $I\!I_J$, $I\!I\!I_J$ of $[J]$ instead of $[J]$. Thus, dependence of the dependent variables in the constitutive theories on $[J]$ must be replaced with their dependence on I_J, $I\!I_J$ and $I\!I\!I_J$. Furthermore, we note that

$$\begin{aligned}
&[\dot{J}] = [L][J] \qquad ; \qquad [D] = \frac{1}{2}\big([L]+[L]^T\big) \qquad ; \qquad [W] = \frac{1}{2}\big([L]-[L]^T\big) \\
\therefore \qquad &[L] = [D]+[W] \qquad \text{hence} \qquad [\dot{J}] = \big([D]+[W]\big)[J]
\end{aligned} \tag{2.28}$$

Thus, dependence on $[\dot{J}]$ can be replaced by the dependence on I_J, $I\!I_J$, $I\!I\!I_J$, $[D]$ and $[W]$. But $[W]$ is pure rotation and hence dependence on $[W]$ can be eliminated. Thus, we can conclude that Helmholtz free energy density must have dependence on I_J, $I\!I_J$, $I\!I\!I_J$ and θ, and the stress tensor and heat vector must have dependence on I_J, $I\!I_J$, $I\!I\!I_J$, $[D]$, θ and $\boldsymbol{g}$.

2.4.2 Eulerian description

The dependence of Helmholtz free energy density, Cauchy stress tensor and heat vector on I_J, $I\!I_J$, $I\!I\!I_J$, though it satisfies the axiom of frame invariance, it is still not so useful due to the fact that $[J]$ and hence I_J, $I\!I_J$, $I\!I\!I_J$ are not deterministic in the Eulerian description. Thus dependence on I_J, $I\!I_J$, $I\!I\!I_J$ must be replaced by some related measures that are obtainable or defined in the Eulerian description. We note the following from conservation of mass

$$\rho_0 = |J|\bar{\rho} = I\!I\!I_J\bar{\rho} \qquad \text{or} \qquad I\!I\!I_J = \frac{\rho_0}{\bar{\rho}} \tag{2.29}$$

in which ρ_0 is density in the reference configuration (hence, constant). Thus dependence on $I\!I\!I_J$ can be replaced with dependence on $1/\bar{\rho}$ or simply $\bar{\rho}$ in the arguments of the Helmholtz free energy density, Cauchy stress tensor and heat vector. I_J and $I\!I_J$ still remain arguments of the Helmholtz free energy density, deviatoric Cauchy stress tensor and heat vector. We note the following.

(1) In Eulerian description $I\!I\!I_J$ must be replaced by $\bar{\rho}$, and I_J, $I\!I_J$ can not be considered in the development of the constitutive theories for the deviatoric Cauchy stress tensor and heat vector due to the fact that I_J and $I\!I_J$ are dependent on the components of $[J]$ which is not deterministic (or obtainable) as the material particle displacements are not known in the Eulerian description. Thus, we conclude that the deviatoric Cauchy stress tensor and heat vector have density $\bar{\rho}$, symmetric part of the velocity gradient tensor $[\bar{D}]$, temperature $\bar{\theta}$ and temperature gradient $\bar{\boldsymbol{g}}$ as their argument tensors in the Eulerian description.

(2) In the Eulerian description we have three choices: contravariant basis, covariant basis and the Jaumann rates, and hence contravariant Cauchy stress tensor $[\bar{\sigma}^{(0)}]$, covariant Cauchy stress tensor $[\bar{\sigma}_{(0)}]$ and the Jaumann stress tensor $[{}^{(0)}\bar{\sigma}^J]$ are obvious choices for measures of stress in the constitutive theory.

(3) Recalling the derivations of the convected time derivatives of the Green's strain tensor in the covariant basis, we note that $[\bar{D}]$ is the convected time derivative of order one of the Green's

strain in the covariant basis, i.e.

$$[\bar{D}] = [\gamma_{(1)}] \tag{2.30}$$

$[\gamma_{(1)}]$ is a fundamental kinematic tensor in covariant basis based on Green's strain tensor, a covariant measure of finite strain.

(4) Likewise if we consider the convected time derivatives of the Almansi strain tensor in contravariant basis, we note that $[\bar{D}]$ is also the convected time derivative of order one of the Almansi strain in contravariant basis, i.e.

$$[\bar{D}] = [\gamma^{(1)}] \tag{2.31}$$

$[\gamma^{(1)}]$ is a fundamental kinematic tensor in contravariant basis derived using the Almansi strain tensor, a contravariant measure of finite strain.

(5) Thus

$$[{}^{(1)}\gamma^J] = [\gamma^{(1)}] = [\gamma_{(1)}] = [\bar{D}] \tag{2.32}$$

holds. That is, the first convected time derivative of the Jaumann strain is also a fundamental kinematic tensor.

(6) We have seen that the convected time derivatives of order higher than one of the Green's strain tensor, the Almansi strain tensor as well as higher order Jaumann rates of strain can be derived in covariant, contravariant and the Jaumann bases which are *fundamental symmetric kinematic tensors of rank two* of various orders in the respective bases. Thus we have

$$[\gamma^{(j)}] \quad ; \quad j = 1, 2, \dots, n \tag{2.33}$$

$$[\gamma_{(j)}] \quad ; \quad j = 1, 2, \dots, n \tag{2.34}$$

$$[{}^{(j)}\gamma^J] \quad ; \quad j = 1, 2, \dots, n \tag{2.35}$$

Hence, instead of considering $[\bar{D}]$ ($[\gamma^{(1)}]$ or $[\gamma_{(1)}]$ or $[{}^{(1)}\gamma^J]$) as argument tensor in the constitutive theories, we can generalize this choice of $[\bar{D}]$ by replacing $[\bar{D}]$ with $[\gamma^{(j)}]$; $j = 1, 2, \dots, n$ or $[\gamma_{(j)}]$; $j = 1, 2, \dots, n$ or $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$ depending upon whether the basis is contra- or co-variant or whether we are considering the Jaumann rates.

(7) In addition to the convected time derivatives of the Green's strain tensor and Almansi strain tensor in co- and contra- variant bases and Jaumann strain rates, we also have convected time derivatives of the contravariant and covariant deviatoric Cauchy stress tensors and Jaumann stress rates in the respective bases (for incompressible as well as compressible matter [1,55]).

$$[{}_d\bar{\sigma}^{(k)}] \quad ; \quad k = 0, 1, \dots, m \tag{2.36}$$

$$[{}_d\bar{\sigma}_{(k)}] \quad ; \quad k = 0, 1, \dots, m \tag{2.37}$$

$$[{}^{(k)}_{\ d}\bar{\sigma}^J] \quad ; \quad k = 0, 1, \dots, m \tag{2.38}$$

These are *fundamental symmetric tensors of rank two*.

(8) Thus, in the development of the rate constitutive theories, in addition to the stress tensors $[{}_d\bar{\sigma}^{(0)}]$, $[{}_d\bar{\sigma}_{(0)}]$ and $[{}^{(0)}_{\ d}\bar{\sigma}^J]$, we must also consider the conjugate pairs of the convected time derivatives of the stress and strain tensors in contra- and co- variant and Jaumann bases.

$$[{}_d\bar{\sigma}^{(k)}] \quad ; \quad k = 0, 1, \dots, m \quad \text{and} \quad [\gamma^{(j)}] \quad ; \quad j = 1, 2, \dots, n \tag{2.39}$$

$$[{}_d\bar{\sigma}_{(k)}] \quad ; \quad k = 0, 1, \dots, m \quad \text{and} \quad [\gamma_{(j)}] \quad ; \quad j = 1, 2, \dots, n \tag{2.40}$$

$$[{}^{(k)}_{\ d}\bar{\sigma}^J] \quad ; \quad k = 0, 1, \dots, m \quad \text{and} \quad [{}^{(j)}\gamma^J] \quad ; \quad j = 1, 2, \dots, n \tag{2.41}$$

This provides many choices and possibilities for developing various constitutive theories in contra- and co- variant and Jaumann bases depending upon the choices of the conjugate convected time derivatives of the stress and strain tensors.

(9) In addition, we also need to consider constitutive theories for $\bar{\boldsymbol{q}}^{(0)}$, $\bar{\boldsymbol{q}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{q}}^J$ in contravariant, covariant and Jaumann bases.

(10) In order to make the derivations and presentation of the details of rate constitutive theories in various bases compact, we introduce the following notations:

Let $[{}^{(k)}_{\;d}\bar{\sigma}]$ *;* $k = 0, 1, \dots, m$ *and* $[{}^{(j)}\gamma]$ *;* $j = 1, 2, \dots, n$ *be the convected time derivatives of the deviatoric stress and strain tensors of orders* $k = 0, 1, \dots, m$ *and* $j = 1, 2, \dots, n$ *in the chosen basis. Likewise, we choose* ${}^{(0)}\bar{\boldsymbol{q}}$ *as the heat vector which becomes* $\bar{\boldsymbol{q}}^{(0)}$*,* $\bar{\boldsymbol{q}}_{(0)}$ *and* ${}^{(0)}\bar{\boldsymbol{q}}^J$ *in contravariant, covariant and Jaumann bases. Furthermore, we let*

$$[{}^{(0)}\bar{\sigma}] = [{}^{(0)}_{\;e}\bar{\sigma}] + [{}^{(0)}_{\;d}\bar{\sigma}] \tag{2.42}$$

be the decomposition of the Cauchy stress tensor $[{}^{(0)}\bar{\sigma}]$ *into equilibrium stress tensor* $[{}^{(0)}_{\;e}\bar{\sigma}]$ *and the deviatoric Cauchy stress tensor* $[{}^{(0)}_{\;d}\bar{\sigma}]$*.*

Using $[{}^{(k)}_{\;d}\bar{\sigma}]$; $k = 0, 1, \dots, m$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ as the convected time derivatives of the deviatoric Cauchy stress and conjugate strain tensors and ${}^{(0)}\bar{\boldsymbol{q}}$ as the heat vector in a chosen basis, and using (2.42), we consider development of the rate constitutive theories for the deviatoric stress tensor, equilibrium stress tensor and heat vector. By choosing $[{}^{(k)}_{\;d}\bar{\sigma}]$; $k = 0, 1, \dots, m$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ as $\big([{}_d\bar{\sigma}^{(k)}]$; $k = 0, 1, \dots, m$, $\bar{\boldsymbol{q}}^{(0)}$ and $[\gamma^{(j)}]$; $j = 1, 2, \dots, n\big)$ or $\big([{}_d\bar{\sigma}_{(k)}]$; $k = 0, 1, \dots, m$, $\bar{\boldsymbol{q}}_{(0)}$ and $[\gamma_{(j)}]$; $j = 1, 2, \dots, n\big)$ or $\big([{}^{(k)}_{\;d}\bar{\sigma}^J]$; $k = 0, 1, \dots, m$, ${}^{(0)}\bar{\boldsymbol{q}}^J$ and $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n\big)$ we can easily obtain various details of rate theories in contravariant basis, covariant basis and using Jaumann rates.

(11) Decomposition (2.42) is necessitated by the entropy inequality when $[\dot{J}]$ is an argument tensor. It is only after (2.42) that the equilibrium stress becomes deterministic using the conditions resulting from the entropy inequality.

2.5 The choices of the dependent variables and their argument tensors in the rate constitutive theories [1, 55–57]

We consider some possible choices of the dependent variables in the rate constitutive theories as well as their argument tensors using the new notation in (2.42) and thereby incorporating all three bases. We consider homogeneous, isotropic and compressible matter unless stated otherwise.

2.5.1 Choice I

The simplest possible choice of the dependent variables in the constitutive theories is of course, the deviatoric Cauchy stress tensor, heat vector and Helmholtz free energy density with their arguments tensors as density $\bar{\rho}$, convected time derivatives of the conjugate strain tensor upto order 'n', temperature $\bar{\theta}$ and temperature gradient $\bar{\boldsymbol{g}}$. Thus, we have the following:

$$\bar{\Phi} = \bar{\Phi}\big(\, \bar{\rho}(\bar{\boldsymbol{x}}, t) \;,\; \bar{\theta}(\bar{\boldsymbol{x}}, t) \,\big) \tag{2.43}$$

$$\big[{}^{(0)}\bar{\sigma}\big] = \big[{}^{(0)}_{\;e}\bar{\sigma}\big(\, \bar{\rho}(\bar{\boldsymbol{x}}, t) \;,\; \bar{\theta}(\bar{\boldsymbol{x}}, t) \,\big)\big] + \big[{}^{(0)}_{\;d}\bar{\sigma}\big] \tag{2.44}$$

$$\big[{}^{(0)}_{\;d}\bar{\sigma}\big] = \big[{}^{(0)}_{\;d}\bar{\sigma}\big(\, \bar{\rho}(\bar{\boldsymbol{x}}, t) \;,\; [{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)] \;;\; j = 1, 2, \ldots, n \;,\; \bar{\theta}(\bar{\boldsymbol{x}}, t) \;,\; \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t) \,\big)\big] \tag{2.45}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\, \bar{\rho}(\bar{\boldsymbol{x}}, t) \;,\; [{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)] \;;\; j = 1, 2 \ldots, n \;,\; \bar{\theta}(\bar{\boldsymbol{x}}, t) \;,\; \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t) \,\big) \tag{2.46}$$

$$[{}_{e}\sigma^{*}]^{T} = \rho_{0}(\boldsymbol{x}) \frac{\partial \Phi\big(\, \rho(\boldsymbol{x}, t) \;,\; \theta(\boldsymbol{x}, t) \,\big)}{\partial [J(\boldsymbol{x}, t)]} \tag{2.47}$$

$[{}^{(0)}_{\;e}\bar{\sigma}]$ in (2.44) needs to be derived using (2.47). Equations (2.43) - (2.47) hold for compressible matter. If the matter is incompressible then $\bar{\rho} = \rho = \rho_{0} =$ constant and thus $\bar{\rho}$ and ρ drop out from the arguments of quantities in (2.43) - (2.47). The choice of 'n' depends upon the desired physics in the constitutive theory. The rate constitutive theories based on (2.43) - (2.47) are termed *ordered rate constitute theories of order 'n'*.

2.5.2 Choice II

We could also consider convected derivative of order 'm' of the deviatoric Cauchy stress tensor as a dependent variable in the constitutive theories in addition to the heat vector and Helmholtz free energy density. We consider density $\bar{\rho}$, convected time derivatives of the chosen strain tensor upto order 'n', temperature $\bar{\theta}$, temperature gradient $\bar{\boldsymbol{g}}$ and the convected time derivatives of the deviatoric Cauchy stress tensor of upto order '$m-1$' as argument tensors of the dependent variables in the constitutive theory. Thus we have the following for the dependent variables in the constitutive theories and their argument tensors.

$$\bar{\Phi} = \bar{\Phi}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ \big) \tag{2.48}$$

$$[{}^{(0)}\bar{\sigma}] = \big[{}^{(0)}_{\ e}\bar{\sigma}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ \big)\big] + \big[{}^{(0)}_{\ d}\bar{\sigma}\big] \tag{2.49}$$

$$\begin{aligned}[{}^{(m)}_{\ d}\bar{\sigma}] = \big[{}^{(m)}_{\ d}\bar{\sigma}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(k)}_{\ d}\bar{\sigma}(\bar{\boldsymbol{x}},t)]\ ;\ k = 0,1,\dots,m-1\ ,\\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j = 1,2,\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\ \big)\big]\end{aligned} \tag{2.50}$$

$$\begin{aligned}{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(k)}_{\ d}\bar{\sigma}(\bar{\boldsymbol{x}},t)]\ ;\ k = 0,1,\dots,m-1\ ,\\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j = 1,2,\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\ \big)\end{aligned} \tag{2.51}$$

$$[{}_{e}\sigma^{*}]^{T} = \rho_{0}(\boldsymbol{x})\frac{\partial\Phi\big(\ \rho(\boldsymbol{x},t)\ ,\ \theta(\boldsymbol{x},t)\ \big)}{\partial[J(\boldsymbol{x},t)]} \tag{2.52}$$

$[{}^{(0)}_{\ e}\bar{\sigma}]$ in (2.49) needs to be derived using (2.52). Equations (2.48) - (2.52) hold for compressible matter. If the matter is incompressible then $\bar{\rho} = \rho = \rho_0 =$ constant and thus $\bar{\rho}$ and ρ drop out from the arguments of quantities in (2.48) - (2.52). Depending upon the choices of 'm' and 'n' in the development of the constitutive theories, varieties of possibilities exist for incorporating diverse physics. The rate constitutive theories based on (2.48) - (2.52) are termed *ordered rate constitute theories of orders* (m,n).

2.5.3 Choice III

In this choice of dependent variables in the rate constitutive theory, we consider the first convected time derivative of the Cauchy stress tensor, the heat vector and Helmholtz free energy density as dependent variables in the development of the constitutive theories. The argument tensors of Helmholtz free energy density are density $\bar{\rho}$ and temperature $\bar{\theta}$. The argument tensors of the first convected time derivative of the deviatoric Cauchy stress tensor and the heat vector are $\bar{\rho}$, $\bar{\theta}$, temperature gradient $\bar{\boldsymbol{g}}$ and the convected time derivatives of the conjugate strain tensor upto orders 'n'. Thus we have the following for the dependent variables in the constitutive theories and their argument tensors:

$$\bar{\Phi} = \bar{\Phi}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ \big) \tag{2.53}$$

$$[{}^{(0)}\bar{\sigma}] = \big[{}^{(0)}_{\ e}\bar{\sigma}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ \big)\big] + \big[{}^{(0)}_{\ d}\bar{\sigma}\big] \tag{2.54}$$

$$[{}^{(1)}_{\ d}\bar{\sigma}] = \big[{}^{(1)}_{\ d}\bar{\sigma}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j=1,2,\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\ \big)\big] \tag{2.55}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j=1,2\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\ \big) \tag{2.56}$$

$$[{}_{e}\sigma^{*}]^{T} = \rho_{0}(\boldsymbol{x})\frac{\partial\Phi\big(\ \rho(\boldsymbol{x},t)\ ,\ \theta(\boldsymbol{x},t)\ \big)}{\partial[J(\boldsymbol{x},t)]} \tag{2.57}$$

$[{}^{(0)}_{\ e}\bar{\sigma}]$ in (2.54) needs to be derived using (2.57). Equations (2.53) - (2.57) hold for compressible matter. If the matter is incompressible then $\bar{\rho} = \rho = \rho_{0}$ = constant and thus $\bar{\rho}$ and ρ drop out from the arguments of quantities in (2.53) - (2.57). The choice of 'n' depends upon the desired physics in the constitutive theory. The rate constitutive theories based on (2.53) - (2.57) are termed *ordered rate constitute theories of orders* $(1,n)$.

Remarks:

(1) The constitutive theories based on (2.43) - (2.47) will be referred to as *ordered rate constitutive theories* due to the fact that they utilize strain rates of various orders. The highest order (say 'n') of the convected time derivative of the strain tensor defines the order of the

rate theory. Thus, 'n' order rate constitutive theories based on (2.43) - (2.47) will contain convected time derivatives of upto orders 'n' of the strain tensor as arguments of the dependent variables in the constitutive theory. Using (2.43) - (2.47) it is possible to derive more complete theories for *thermofluids*. Newtonian fluids and generalized Newtonian fluids such as power law, Carreau-Yasuda fluids etc are a subset of the constitutive theories resulting from this approach. These theories will be considered in chapter 4.

(2) The constitutive theories based on (2.48) - (2.52) are also referred to as *ordered rate constitutive theories* but these theories are of orders (m, n) in stress and strain rate tensors, in which 'm' and 'n' define the highest orders of the convected time derivatives of the conjugate stress and strain tensors used in the development of the constitutive theory. Using (2.48) - (2.52), it is possible the derive more complete rate constitutive theories for *polymeric fluids*. Maxwell model, Oldroyd-B model, Giesekus model etc. for dilute and dense polymeric fluids (both compressible and incompressible) are a subset of the constitutive theories resulting from this approach. We consider derivations of these theories in chapter 5.

(3) In choice III, *ordered rate constitutive theories* can be derived using (2.53) - (2.57) for *thermoelastic solids*. Hypo-thermoelastic solids and generalized hypo-thermoelastic solids are examples of the rate constitutive theories resulting as a subset of this approach. These rate theories will be considered in chapter 3.

(4) We note that equilibrium stress tensor ${}_{e}^{(0)}\bar{\boldsymbol{\sigma}}$ is strictly deterministic from the Helmholtz free energy density regardless of choice I, II or III for the dependent variables in the constitutive theories and their argument tensors.

(5) For the deviatoric Cauchy stress tensor, the entropy inequality does not provide means of establishing the constitutive theories but requires that the work expanded due to the deviatoric Cauchy stress tensor be positive. We shall consider the *theory of generators and invariants* for deriving rate constitutive theories for the deviatoric Cauchy stress tensor for all three choices.

(6) The theory of generators and invariants also provides a more comprehensive development for the constitutive theories for the heat vector and will be considered in the present work. Details are presented in the following chapters.

2.6 Constitutive theory for equilibrium stress tensor: compressible matter [1,55,56]

We recall that from the entropy inequality in Lagrangian description, we have the following for the first Piola-Kirchhoff equilibrium stress tensor $[{}_e\sigma^*]$.

$$[{}_e\sigma^*]^T = \rho_0(\boldsymbol{x}) \frac{\partial \Phi\big(\,\rho(\boldsymbol{x},t), \theta(\boldsymbol{x},t)\,\big)}{\partial [J(\boldsymbol{x},t)]} \tag{2.58}$$

Consider equilibrium stress tensor $[{}_e\bar{\sigma}^{(0)}]$ in contravariant basis. For compressible matter

$$[{}_e\bar{\sigma}^{(0)}] = |J|^{-1}[{}_e\sigma^*]^T[J]^T \tag{2.59}$$

and we obtain the following by substituting from (2.58) into (2.59)

$$[{}_e\bar{\sigma}^{(0)}] = |J|^{-1}\rho_0 \frac{\partial \bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial [J]}[J]^T \tag{2.60}$$

Further simplification of (2.60) requires determination of $\frac{\partial \bar{\Phi}}{\partial [J]}$. We recall that we had considered $\Phi = \Phi([J],\theta)$ and then replaced $[J]$ by $I\!I\!I_J = |J|$ as Φ must be frame invariant (and neglected dependence of Φ on I_J, $I\!I_J$). Since $[J]$ is not deterministic in Eulerian description, I_J, $I\!I_J$ cannot be considered as arguments of Φ and we use continuity $\rho_0 = |J|\bar{\rho}$ to express $|J| = \rho_0/\bar{\rho}$. Thus, $\bar{\Phi} = \bar{\Phi}(\rho_0/\bar{\rho},\bar{\theta})$ is fundamental in all further developments. Since ρ_0 is the density in the reference configuration (constant) we can write $\bar{\Phi} = \bar{\Phi}(1/\bar{\rho},\bar{\theta}) = \bar{\Phi}(\bar{v},\bar{\theta})$, where $\bar{v}$ is specific volume in the current configuration. We can also consider $\bar{\bar{\Phi}} = \bar{\bar{\Phi}}(\bar{\rho},\bar{\theta})$. Thus for deriving details of $\frac{\partial \bar{\Phi}}{\partial [J]}$ we have two possible approaches. In the first approach we consider $\bar{\Phi} = \bar{\Phi}(\bar{\rho},\bar{\theta})$ whereas in the second approach we consider $\bar{\Phi} = \bar{\Phi}(\bar{v}, \bar{\theta})$. Details are presented in the following.

First approach: Consider $\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta})$

In this case we have

$$\frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial [J]} = \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} \frac{\partial \bar{\rho}}{\partial |J|} \frac{\partial |J|}{\partial [J]} \tag{2.61}$$

From conservation of mass

$$\rho_0 = \bar{\rho}|J| \qquad \text{or} \qquad \bar{\rho} = \frac{\rho_0}{|J|} \tag{2.62}$$

$$\therefore \qquad \frac{\partial \bar{\rho}}{\partial |J|} = -\frac{\rho_0}{|J|^2} = -\frac{\bar{\rho}}{|J|} = -\frac{\bar{\rho}^2}{\rho_0} \tag{2.63}$$

$$\text{and} \qquad \frac{\partial |J|}{\partial [J]} = [J^{-1}]^T |J| \tag{2.64}$$

Substituting from (2.63) and (2.64) into (2.61)

$$\frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial [J]} = \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} \Big(-\frac{\bar{\rho}^{\,2}}{\rho_0} \Big) [J^{-1}]^T |J| \tag{2.65}$$

Substituting from (2.65) into (2.60)

$$[{}_e\bar{\sigma}^{(0)}] = |J|^{-1} \rho_0 \Big(\frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} \Big(-\frac{\bar{\rho}^{\,2}}{\rho_0} \Big) [J^{-1}]^T \Big) [J]^T |J| \tag{2.66}$$

$$\text{or} \qquad [{}_e\bar{\sigma}^{(0)}] = -\bar{\rho}^{\,2} \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} [J^{-1}]^T [J]^T \tag{2.67}$$

Since $[J^{-1}]^T [J]^T = \big([J][J]^{-1} \big)^T = [I]^T = [I]$, we can write the following for $[{}_e\bar{\sigma}^{(0)}]$

$$[{}_e\bar{\sigma}^{(0)}] = -\bar{\rho}^{\,2} \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} [I] \tag{2.68}$$

and if we let

$$\bar{p}(\bar{\rho}, \bar{\theta}) = -\bar{\rho}^{\,2} \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} \tag{2.69}$$

then

$$[{}_e\bar{\sigma}^{(0)}] = \bar{p}(\bar{\rho}, \bar{\theta})[I] \tag{2.70}$$

Second approach: Consider $\bar{\Phi} = \bar{\Phi}(\bar{v}, \bar{\theta})$

In this case

$$\frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial [J]} = \frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial \bar{v}} \frac{\partial \bar{v}}{\partial |J|} \frac{\partial |J|}{\partial [J]} \tag{2.71}$$

Since $\frac{1}{\bar{\rho}} = \bar{v} = \frac{|J|}{\rho_0}$, we have

$$\frac{\partial \bar{v}}{\partial |J|} = \frac{1}{\rho_0} \tag{2.72}$$

Substituting from (2.72) and (2.64) into (2.71)

$$\frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial [J]} = \frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial \bar{v}} \frac{1}{\rho_0} [J^{-1}]^T |J| \tag{2.73}$$

Using (2.73) in (2.60)

$$[{}_e\bar{\sigma}^{(0)}] = |J|^{-1} \rho_0 \left(\frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial \bar{v}} \frac{1}{\rho_0} [J^{-1}]^T \right) [J]^T |J| \tag{2.74}$$

or $$[{}_e\bar{\sigma}^{(0)}] = \frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial \bar{v}} \big[[J][J]^{-1} \big]^T = \frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial \bar{v}} [I] \tag{2.75}$$

and if we let

$$\bar{p}(\bar{v}, \bar{\theta}) = \frac{\partial \bar{\Phi}(\bar{v}, \bar{\theta})}{\partial \bar{v}} \tag{2.76}$$

then

$$[{}_e\bar{\sigma}^{(0)}] = \bar{p}(\bar{v}, \bar{\theta})[I] \tag{2.77}$$

Both definitions of $p(\cdot)$ (equations (2.69) and (2.76)) are admissible.

***Remarks*:**

1. We note that (2.69) can be derived using (2.76) and vice-versa.

$$\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = \frac{\partial\bar{\Phi}}{\partial\bar{v}}\frac{\partial\bar{v}}{\partial\bar{\rho}} = \frac{\partial\bar{\Phi}}{\partial\bar{v}}\Big(-\frac{1}{\bar{\rho}^{\,2}}\Big) \tag{2.78}$$

$$\therefore \qquad \frac{\partial\bar{\Phi}}{\partial\bar{v}} = -\bar{\rho}^{\,2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} \tag{2.79}$$

Substituting from (2.79) in (2.76)

$$\bar{p}(\bar{v},\bar{\theta}) = -\bar{\rho}^{\,2}\frac{\partial\bar{\Phi}}{\partial\bar{\rho}} = \bar{p}(\bar{\rho},\bar{\theta}) \tag{2.80}$$

which is same as (2.69).

2. $\bar{p}(\bar{\rho},\bar{\theta})$ or $\bar{p}(\bar{v},\bar{\theta})$ is *thermodynamic pressure*. From (2.70) and (2.77), we clearly note that $[\,{}_e\bar{\sigma}^{(0)}]$ is independent of basis, i.e., $[\,{}_e\bar{\sigma}_{(0)}] = [\,{}_e\bar{\sigma}^{(0)}] = [\,{}^{(0)}_{e}\bar{\sigma}^{J}]$.

3. Thus, for compressible matter regardless of the choice of basis we have the following for all three choices of the dependent variables and their argument tensors.

$$[\,{}^{(0)}\bar{\sigma}] = \bar{p}(\,\bar{\rho}\,,\,\bar{\theta}\,)[I] + [\,{}^{(0)}_{d}\bar{\sigma}] \qquad \text{or} \qquad [\,{}^{(0)}\bar{\sigma}] = \bar{p}(\,\bar{v}\,,\,\bar{\theta}\,)[I] + [\,{}^{(0)}_{d}\bar{\sigma}] \tag{2.81}$$

4. From (2.70) and (2.77), we clearly note that a rigid rotation of $\bar{x}_i$ coordinates to $\bar{x}'_i$ does not alter $[{}_e\bar{\sigma}^{(0)}]$. Obviously similar expressions hold true for $[{}_e\bar{\sigma}_{(0)}]$ and $[{}^{(0)}_{e}\bar{\sigma}^{J}]$ also.

5. The thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})$ or $\bar{p}(\bar{v},\bar{\theta})$ is completely deterministic from the deformation field once the Helmholtz free energy density is defined and is known as *equation of state*. Instead of $\bar{p}(\bar{\rho},\bar{\theta})$ or $\bar{p}(\bar{v},\bar{\theta})$, we can use $-\bar{p}(\bar{\rho},\bar{\theta})$ or $-\bar{p}(\bar{v},\bar{\theta})$ in (2.81) if we define compressive pressure to be positive.

6. From $\bar{p}(\bar{\rho},\bar{\theta})$ in (2.69) and $\bar{p}(\bar{v},\bar{\theta})$ in (2.76), we note that equation of state can be defined either using $\bar{\rho}$, $\bar{\theta}$ or $\bar{v}$, $\bar{\theta}$ as one form can be transformed into the other using $\bar{v} = 1/\bar{\rho}$.

2.7 Constitutive theory for equilibrium stress tensor: incompressible matter [1,55,56]

We recall that from the entropy inequality in Lagrangian description, we have the following for the first Piola-Kirchhoff equilibrium stress tensor $[{}_e\sigma^*]$.

$$[{}_e\sigma^*]^T = \rho_0(\boldsymbol{x})\frac{\partial\Phi\big(\,\rho(\boldsymbol{x},t),\theta(\boldsymbol{x},t)\,\big)}{\partial[J(\boldsymbol{x},t)]} \tag{2.82}$$

For incompressible matter $\bar{\rho} = \rho = \rho_0$ which implies that $|J| = 1$, hence $\Phi = \Phi(\theta)$ and therefore $\frac{\partial\Phi}{\partial[J]} = 0$ (due to (2.61)). Consider $[\,{}_e\bar{\sigma}^{(0)}]$, equilibrium stress in the contravariant basis. In this case $[\,{}_e\bar{\sigma}^{(0)}]$ can not be determined using the derivation considered for the compressible case in section 2.6. Instead, the *incompressibility condition* $|J| = 1$ must be enforced. We note that $\bar{\rho} = \rho = \rho_0$ also implies that in Eulerian description $\text{tr}([\bar{D}]) = 0$ must hold. Hence for incompressible matter

$$\text{tr}([\bar{D}]) = \text{tr}([\bar{L}]) = \text{tr}([\dot{J}][J]^{-1}) = \dot{J}_{ik}(J^{-1})_{ki} = 0 \tag{2.83}$$

We enforce (2.83) through entropy inequality. If (2.83) holds then

$$p\,\dot{J}_{ik}(J^{-1})_{ki} = p(\theta)\dot{J}_{ik}(J^{-1})_{ki} = 0 \tag{2.84}$$

must also hold, where p is a *Lagrange multiplier*. p cannot be a function of the Jacobian but can depend upon temperature i.e. $p(\theta)$ is valid. We add (2.84) to the left hand side of entropy inequality (2.17) (since (2.84) is zero, it does not change the meaning of entropy inequality) and decompose $[\sigma*]$ into equilibrium and deviatoric parts.

$$\Big(\rho_0\frac{\partial\Phi}{\partial J_{ik}} - {}_e\sigma^*_{ki}\Big)\dot{J}_{ik} - {}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J|\underset{\sim}{q_i}\underset{\sim}{g_i}}{\theta} + p(\theta)\dot{J}_{ik}(J^{-1})_{ki} \leq 0 \tag{2.85}$$

Regrouping the terms and substituting $\rho_0 \frac{\partial \Phi}{\partial J_{ik}} = 0$ in (2.85) gives

$$\Big(p(\theta)(J^{-1})_{ki} - {}_e\sigma^*_{ik}\Big)\dot{J}_{ik} - {}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J|\underset{\sim}{q_i}\underset{\sim}{g_i}}{\theta} \leq 0 \tag{2.86}$$

Following the same reasoning as in section 2.3 (inequalities (2.19) and (2.20)), then

$${}_d\sigma^*_{ki}\dot{J}_{ik} > 0 \tag{2.87}$$

$$\text{and} \qquad {}_e\sigma^*_{ki} = p(\theta)(J^{-1})_{ki} \tag{2.88}$$

Equation (2.88) can also be written as

$$[{}_e\sigma^*]^T = p(\theta)[J^T]^{-1} \tag{2.89}$$

but the contravariant Cauchy stress tensor and the first Piola-Kirchhoff stress tensor are related

$$[\bar{\sigma}^{(0)}] = |J|^{-1}[\sigma^*]^T[J]^T \tag{2.90}$$

and $|J| = 1$ in this case due to incompressibility, hence

$$[\bar{\sigma}^{(0)}] = [\sigma^*]^T[J]^T \tag{2.91}$$

Post multiply (2.89) by $[J]^T$

$$[{}_e\sigma^*]^T[J]^T = \bar{p}(\bar{\theta})[J^T]^{-1}[J]^T = \bar{p}(\bar{\theta})[I] \tag{2.92}$$

and by using (2.91) for the left side of (2.92) we obtain

$$[{}_e\bar{\sigma}^{(0)}] = \bar{p}(\bar{\theta})[I] \tag{2.93}$$

***Remarks*:**

1. $\bar{p}(\bar{\theta})$ is called *mechanical pressure*. It is obvious that $\bar{p}(\bar{\theta})$ is not deterministic from the deformation field as it is an arbitrary Lagrange multiplier but can be a function of temperature $\bar{\theta}$.

2. Instead of $\bar{p}(\bar{\theta})$, we can use $-\bar{p}(\bar{\theta})$ in (2.93), if we define compressive pressure to be positive.

3. In this case also, as in compressible case, $[{}_e\bar{\sigma}^{(0)}]$ is not affected by a rigid rotation of $\bar{x}_i$-coordinates in the current configuration to $\bar{x}'_i$. Thus, we have

$$[{}_e\bar{\sigma}^{(0)}] = [{}_e\bar{\sigma}_{(0)}] = [{}^{(0)}_{\;e}\bar{\sigma}] = \bar{p}(\bar{\theta})[I] \tag{2.94}$$

   That is, mechanical pressure $p(\bar{\theta})$ in all three bases is the same.

2.8 Theory of invariants and generators [1,3–21]

From the entropy inequality and the Cauchy stress decomposition and the derivations presented in section 2.6, we note that the equilibrium Cauchy stress tensor is deterministic from the conditions resulting from the entropy inequality for compressible matter as well as incompressible matter (by using incompressibility constraint) but the deviatoric Cauchy stress tensor is not. The conditions resulting from the entropy inequality require that the work expanded due to the deviatoric Cauchy stress tensor be positive but the entropy inequality provides no further mechanism for developing constitutive theories for the deviatoric Cauchy stress tensor. The theory of invariants and the generators can be utilized for establishing constitutive theories for the deviatoric Cauchy stress tensor. Additionally this approach also provides more general constitutive theories for the deviatoric Cauchy stress tensor as well as the heat vector.

When using the theory of generators and invariants [1, 3–21] to derive constitutive theories for dependent variables in the constitutive theories we must follow certain rules and guidelines to ensure that axioms of the constitutive theories are not violated. See references [1, 55–57] for complete details. A summary is presented in the following

(1) The arguments of the dependent variables in the constitutive theories must be tensors. These can be of the same rank as the dependent variables or of tanks higher as well as lower.

(2) The argument list may contain symmetric as well as skew symmetric tensors (depending upon the physics).

(3) The combined generators of the argument tensors of the dependent variables must be of the same rank and type as the dependent variables in the constitutive theory, then the combined generators of its argument tensors must also be symmetric tensors of rank two. Likewise, if $\boldsymbol{q}$, a tensor of rank one, is a dependent variable in the constitutive theory, then the combined generators of its argument tensors must also be tensors of rank one. Since the combined generators of the argument tensors form integrity (i.e. minimal basis), we can express the dependent variables in the constitutive theories as a linear combination of their combined generators. Thus, the stress tensor $\boldsymbol{\sigma}$, a symmetric tensor of rank two, and heat vector $\boldsymbol{q}$, a tensor of rank one, can be expressed as

$$[\sigma] = {}^{\sigma}\alpha^0[I] + \sum_{i=1}^{N} {}^{\sigma}\alpha^i[{}^{\sigma}\underset{\sim}{G}^i] \tag{2.95}$$

$$\{q\} = -\sum_{i=1}^{\widetilde{N}} {}^{q}\alpha^i\{{}^{q}\underset{\sim}{G}^i\} \tag{2.96}$$

in which $[{}^{\sigma}\underset{\sim}{G}^i]$ are the combined generators of the argument tensors of $[\sigma]$, symmetric tensor of rank two, and $\{{}^{q}\underset{\sim}{G}^i\}$ are combined generators of the argument tensors of $\{q\}$, tensors of rank one. The coefficients ${}^{\sigma}\alpha^i$ are functions of the combined invariants of the argument tensors of $[\sigma]$. Likewise, ${}^{q}\alpha^i$ are functions of the combined invariants of the argument tensors of $\{q\}$. It is straight forward to show that (2.95) and (2.96) are form invariant and the coefficients ${}^{\sigma}\alpha^i$ and ${}^{q}\alpha^i$ are naturally frame invariant as they are functions of the invariants of their argument tensors. To prove form invariance of (2.95) and (2.96) we consider change of frame from x-frame to x'-frame according to

$$\{x'\} = [R]\{x\} \tag{2.97}$$

then (2.95) and (2.96) transform into (in x'-frame)

$$\begin{aligned}
[\sigma]' &= {}^{\sigma}\alpha^0[I]' + \sum_{i=1}^{N} {}^{\sigma}\alpha^i[{}^{\sigma}\underset{\sim}{G}^i]' \\
\{q\}' &= -\sum_{i=1}^{\widetilde{N}} {}^{q}\alpha^i\{{}^{q}\underset{\sim}{G}^i\}'
\end{aligned} \tag{2.98}$$

in which $[\sigma]'$ and $[{}^{\sigma}\underset{\sim}{G}^i]'$ in x'-frame are obtained from $[\sigma]$ and $[{}^{\sigma}\underset{\sim}{G}^i]$ in x-frame using standard transformation for tensors of rank two.

$$\begin{aligned}
&[\sigma]' = [R][\sigma][R]^T \\
&[{}^{\sigma}\underset{\sim}{G}^i]' = [R][{}^{\sigma}\underset{\sim}{G}^i][R]^T \qquad ; \qquad i = 1, 2, \dots, N \\
&[I]' = [R][I][R]^T = [I]
\end{aligned} \tag{2.99}$$

Likewise $\{q\}'$ and $\{{}^{q}\underset{\sim}{G}^i\}'$ in x'-frame are obtained from $\{q\}$ and $\{{}^{q}\underset{\sim}{G}^i\}$ in x-frame using transformation for tensors of rank one.

$$\begin{aligned}
\{q\}' &= [R]\{q\} \\
\{{}^{q}\underset{\sim}{G}^i\}' &= [R]\{{}^{q}\underset{\sim}{G}^i\}
\end{aligned} \tag{2.100}$$

Thus, (2.95) and (2.96) are form invariant in which the coefficients are frame invariant. These are strict requirements for a valid constitutive theories based on the axioms of the constitutive theory.

(4) If $[s]$ is a symmetric tensor of rank two, $\{v\}$ is a tensor of rank one (i.e. a vector) and $[w]$ is a skew symmetric tensor of rank two, then based on Eringen and references [1, 3–21] we can obtain combined invariants and generators using these three, provided in appendix A.

Table A.1 provides complete and irreducible sets of invariants of $[s]$, $\{v\}$ and $[w]$.

Table A.1 lists the generators of rank one.

Table A.3 gives symmetric generators of rank two.

The skew symmetric generators of rank two are listed in table A.4.

The ordered constitutive theories for thermoelastic solids, thermoviscous fluids and thermoviscoelastic fluids in Eulerian description using the theory of invariants and generators are presented in chapter 3 - 5.

Chapter 3

Rate Constitutive Theories in Eulerian Description for Ordered Thermoelastic Solids

3.1 Introduction

When the mathematical models for the deforming solids are constructed using the Eulerian description, the material particle displacements and hence the strain measures are not known. In such cases the constitutive theory must utilize convected time derivatives of the strain measures. The rate constitutive equations are generally derived using constitutive theories in which the stress rates are functions of the strain rates [2, 22, 29, 30, 40, 42, 55, 56]. The need for such constitutive theories arises when the mathematical models for the deforming solid are constructed using the Eulerian description in which material particles are not followed during the deformation.

The material presented in this chapter focuses on the development of rate constitutive theories for ordered thermoelastic solids for which the mathematical models are in the Eulerian description. We begin all developments with entropy inequality, an essential conservation law for the development of constitutive theories. The Cauchy stress tensor is decomposed into equilibrium stress and deviatoric stress as necessitated by the entropy inequality. The constitute equation for the equilibrium stress for both compressible and incompressible cases is established using entropy inequality [55,56]. For the deviatoric Cauchy stress tensor, the entropy inequality does not provide any mechanism for establishing constitutive theories. For thermoelastic solids, the first convected time derivative of the deviatoric Cauchy stress tensor and the heat vector are expressed as functions of density, temperature, temperature gradient and convected time derivatives up to any desired order 'n' of the conjugate strain tensor. The solids described by these constitutive theories will be referred to as *ordered thermoelastic solids* in which the highest order of the convected time derivative of the strain tensor defines the order of the thermoelastic solid. We use the theory of generators and invariants to (i) establish a most general form of the rate constitutive theories in which the first convected time derivative of the chosen deviatoric Cauchy stress tensor can be a function of the convected time derivatives up to any desired order of the conjugate strain tensor (and other arguments), (ii) specialize the general theory presented in (i) to second order thermoelastic solids, and (iii) further specialize the theory presented in (ii) to first order thermoelastic solids and demonstrate that the general constitutive theory of ordered thermoelastic solids of order one reduces to the well known hypo-elasticity with further assumptions. All derivations and details for (i) - (iii) are presented using contravariant and covariant bases as well as Jaumann rates for incompressible and compressible thermoelastic solids. Discussion and arguments are presented for the validity and usefulness of the contravariant, covariant and well as the validity of Jaumann rate constitutive equations.

3.2 Rate Constitutive Theories in Eulerian Description

In chapter 2 we had considered entropy inequality in Lagrangian description to conclude that Φ, $\boldsymbol{\sigma}^*$, $\boldsymbol{q}$ and η must be the dependent variables in the constitutive theories. We considered $[J]$, $[\dot{J}]$, θ and $\boldsymbol{g}$ as arguments of the dependent variables in the constitutive theories. Using entropy inequality in Lagrangian description it was concluded that: (i) Φ is not a function of $[\dot{J}]$ (ii) Φ is not a function of $\boldsymbol{g}$ either (iii) η is not a dependent variable in the constitutive theory (iv) consideration of $\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - \sigma^*_{ki} = 0$ and $\underset{\sim}{q_i}\underset{\sim}{g_i} \leq 0$ is inappropriate due to the fact that in this case $\boldsymbol{\sigma}^*$ is not a function of $[\dot{J}]$ as Φ is not a function of $[\dot{J}]$, which is contrary to the assumption that $\boldsymbol{\sigma}^*$ depends on $[\dot{J}]$. Thus, entropy inequality does not provide any further means of determining the constitutive theories for neither $\boldsymbol{\sigma}^*$ nor $\boldsymbol{q}$. It was shown that by considering stress decomposition into equilibrium and deviatoric stress i.e. $\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^* + {}_d\boldsymbol{\sigma}^*$ in which ${}_e\boldsymbol{\sigma}^*$ is not a function of $[\dot{J}]$ and ${}_d\boldsymbol{\sigma}^*$ becomes zero when $[\dot{J}]$ and $\boldsymbol{g}$ are zero, and using the conditions resulting from the entropy inequality, that ${}_d\sigma^*_{ki}\dot{J}_{ik} > 0$ and $\underset{\sim}{q_i}\underset{\sim}{g_i} \leq 0$ must hold, which gave us $\Phi = \Phi([J], \theta)$, $\sigma^*_{ij} = \rho_0 \frac{\partial \Phi}{\partial J_{ji}} + {}_d\sigma^*_{ij}([J], [\dot{J}], \theta, \boldsymbol{g})$ and $\boldsymbol{q} = \boldsymbol{q}([J], [\dot{J}], \theta, \boldsymbol{g})$. Due to frame invariance considerations, dependence on $[J]$ must be replaced by I_J, $I\!I_J$, $I\!I\!I_J$ and $[\dot{J}]$ can be replaced by I_J, $I\!I_J$, $I\!I\!I_J$, $[D]$. These arguments hold in Lagrangian description. However, in Eulerian description, material point displacements are not known, hence $[J]$ is not deterministic but $I\!I\!I_J = \det[J] = \rho_0/\bar{\rho}$ i.e. dependence on $I\!I\!I_J$ can be replaced by $\bar{\rho}$, but dependence on I_J and $I\!I_J$ can not be considered. Thus, in Eulerian description we consider the following in contravariant basis

$$
\begin{aligned}
&\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \\
&[\bar{\sigma}^{(0)}] = [\,{}_e\bar{\sigma}^{(0)}\,] + [\,{}_d\bar{\sigma}^{(0)}(\bar{\rho}, [\bar{D}], \bar{\theta}, \bar{\boldsymbol{g}})\,] \quad ; \quad [{}_e\sigma^*]^T = \rho_0 \frac{\partial \Phi}{\partial [J]} \\
&\bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}^{(0)}(\bar{\rho}, [\bar{D}], \bar{\theta}, \bar{\boldsymbol{g}})
\end{aligned}
\tag{3.1}
$$

For compressible matter, equilibrium stress is a function $\bar{\Phi}$ and thus it is deterministic from the deformation field. For incompressible matter, equilibrium stress is also derived from the entropy inequality in conjunction with incompressibility constraint, however, equilibrium stress is not a function of $\bar{\Phi}$ and thus it is not deterministic from the deformation field. It was shown that in both cases, equilibrium stress is independent of the basis. We make the following remarks:

(1) The second law of thermodynamics only restricts the work expanded due to the deviatoric stress to be positive but provides no mechanism for determining the constitutive theory for

the deviatoric stress. In addition, $\underset{\sim}{q}_i \underset{\sim}{g}_i \leq 0$ must also hold.

(2) The theory of generators and invariants [3–21] provides a continuum mechanics foundation to derive constitutive equations for the deviatoric Cauchy stress tensor and heat vector in which we determine combined generators of the argument tensors that form *integrity or minimal basis*. The dependent variables in the constitutive theories are expressed as linear combinations of the combined generators of the argument tensors. The coefficients used in the linear combinations are functions of $\bar{\rho}$, $\bar{\theta}$, and the combined invariants of the argument tensors in the current configuration which, using the *axiom of smooth neighborhood*, are determined by using their Taylor series expansion about a previously known configuration.

3.3 Thermoelastic solids: dependent variables in the constitutive theories and their argument tensors

Let $[\gamma^{(j)}]$, $[\gamma_{(j)}]$, $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$ be the convected time derivatives of order $1, 2, \dots, n$ of the Almansi strain tensor $[\bar{\varepsilon}]$, Green's strain tensor $[\varepsilon]$ and Jaumann strain rates. These are *fundamental kinematic symmetric tensors of rank two*. Likewise, let $[{}_d\bar{\sigma}^{(k)}]$, $[{}_d\bar{\sigma}_{(k)}]$ and $[{}^{(k)}_{\ d}\bar{\sigma}^J]$; $k = 0, 1, \dots, m$ be convected time derivatives of orders $0, 1, \dots, m$ of the Cauchy stress tensors in the three bases. These are *fundamental symmetric tensors of rank two*. We note that $[\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}]$. Using the new notation introduced in chapter 2 we can generalize (3.1) by replacing $[\bar{D}]$ with $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ depending upon whether the basis is contra- or co-variant or Jaumann basis.

Consider current configuration at time $t = t_{n+1}$. In the constitutive theories presented in this chapter for thermoelastic solids, dependence of $[\ {}^{(0)}_{\ d}\bar{\sigma}\ ]$ on $[{}^{(j)}\gamma]$; $j = 0, 1, \dots, n$ (in addition to $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$) is through dependence of $[\ {}^{(1)}_{\ d}\bar{\sigma}\ ]$ on $[{}^{(j)}\gamma]$; $j = 0, 1, \dots, n$. Thus, we consider $\bar{\Phi}(\cdot)$, Helmholtz free energy density; $[{}^{(1)}_{\ d}\bar{\sigma}\ ]$, first convected time derivative of the deviatoric Cauchy stress tensor, and ${}^{(0)}\bar{\boldsymbol{q}}$, the heat vector, as dependent variables in the constitutive theories. $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$, $[{}^{(j)}\gamma\,]$; $j = 1, 2, \dots, n$ appear (as appropriate) as their argument tensors. Hence, we have the following for compressible and incompressible thermoelastic solids (see chapter 2, choice II).

Compressible thermoelastic solids:

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\big) \tag{3.2}$$

$$[{}^{(0)}\bar{\sigma}\,] = \big[{}^{(0)}_{e}\bar{\sigma}\big(\bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\big)\big] + \big[{}^{(0)}_{d}\bar{\sigma}\big] \tag{3.3}$$

$$[{}^{(1)}_{d}\bar{\sigma}\,] = \big[{}^{(1)}_{d}\bar{\sigma}\big(\bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big)\big] \tag{3.4}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j = 1, 2 \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big) \tag{3.5}$$

in which equilibrium stress $[{}^{(0)}_{e}\bar{\sigma}]$ is thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})[I]$ and it is independent of the basis. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\rho},\bar{\theta})$ can be replaced by $-\bar{p}(\bar{\rho},\bar{\theta})$ (see chapter 2, section 2.6 for derivation).

Incompressible thermoelastic solids:

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\theta}(\bar{\boldsymbol{x}},t)\big) \tag{3.6}$$

$$[{}^{(0)}\bar{\sigma}\,] = \big[{}^{(0)}_{e}\bar{\sigma}\big(\bar{\theta}(\bar{\boldsymbol{x}},t)\big)\big] + \big[{}^{(0)}_{d}\bar{\sigma}\big] \tag{3.7}$$

$$[{}^{(1)}_{d}\bar{\sigma}\,] = \big[{}^{(1)}_{d}\bar{\sigma}\big(\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big)\big] \tag{3.8}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j = 1, 2 \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big) \tag{3.9}$$

where equilibrium stress $[{}^{(0)}_{e}\bar{\sigma}]$ is mechanical pressure $\bar{p}(\bar{\theta})[I]$ and it also is independent of the basis. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\theta})$ can be replaced by $-\bar{p}(\bar{\theta})$ (see chapter 2, section 2.6 for derivation).

The rate constitutive theories of various orders for $[{}^{(1)}_{d}\bar{\sigma}\,]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ derived using (3.4) and (3.5), or (3.8) and (3.9) for compressible and incompressible case can be converted to contravariant basis, covariant basis or the Jaumann rates by choosing $[{}^{(k)}_{d}\bar{\sigma}\,]$; $k = 0, 1$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma\,]$; $j = 1, 2, \dots, n$ as $([{}_{d}\bar{\sigma}^{(k)}]$; $k = 0, 1$, $\bar{\boldsymbol{q}}^{(0)}$ and $[\gamma^{(j)}]$; $j = 1, 2, \dots, n)$ or $([{}_{d}\bar{\sigma}_{(k)}]$; $k = 0, 1$, $\bar{\boldsymbol{q}}_{(0)}$ and $[\gamma_{(j)}]$; $j = 1, 2, \dots, n)$ or $([{}^{(k)}_{d}\bar{\sigma}^{J}]$; $k = 0, 1$, ${}^{(0)}\bar{\boldsymbol{q}}^{J}$ and $[{}^{(j)}\gamma^{J}]$; $j = 1, 2, \dots, n)$.

In the following sections we consider details of the derivations of rate constitutive theories in Eulerian description for the deviatoric Cauchy stress tensor and heat vector for both compressible as well as incompressible thermoelastic solids.

3.4 Rate constitutive theory of order '*n*': compressible thermoelastic solids

Consider a deforming volume of compressible thermoelastic solid at time $t = t_{n+1}$, the current configuration. We derive the rate constitutive theory of order 'n' for deviatoric Cauchy stress $[{}^{(0)}_{d}\bar{\sigma}]$ and heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ using ((3.4) and (3.5))

$$
\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] &= [{}^{(1)}_{d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(j)}\gamma]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \\
{}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [{}^{(j)}\gamma]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}\big)
\end{aligned}
\tag{3.10}
$$

3.4.1 Constitutive theory for the deviatoric Cauchy stress tensor

Let $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \dots, N$ be the combined generators of $[{}^{(1)}_{d}\bar{\sigma}]$ of the argument tensors $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two, and let ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, M$ be the combined invariants of the same argument tensors. Then, we can express $[{}_{d}\bar{\sigma}^{(1)}]$ as a linear combination of the generators $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \dots, N$ and the identity tensor $[I]$.

$$
[{}^{(1)}_{d}\bar{\sigma}] = {}^{\sigma}\alpha^{0}[I] + \sum_{i=1}^{N} {}^{\sigma}\alpha^{i}[{}^{\sigma}\underset{\sim}{G}^{i}]
\tag{3.11}
$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, N$ in (3.11) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \ldots, M$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, N$ in (3.11) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, N$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{M} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 0, 1, \ldots, N \quad (3.12)
$$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \ldots, M$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \ldots, N$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$ whereas in (3.12), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\,,\, \bar{\theta}_{t_n}\,,\, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, M\,,\, \bar{\theta}_{t_{n+1}}\,,\, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \ldots, M\big)$; $i = 0, 1, \ldots, N$. When (3.12) is substituted in (3.11), we obtain the final expression for the most general rate constitutive theory of order n for $[{}^{(1)}_{\;d}\bar{\sigma}]$ for compressible thermoelastic solids. This theory uses integrity and hence is complete.

3.4.2 Constitutive theory for the heat vector

Let $\{{}^{q}\underset{\sim}{G}^i\}$; $i = 1, 2, \ldots, \widetilde{N}$ be the combined generators of ${}^{(0)}\bar{\boldsymbol{q}}$ of the argument tensors $[{}^{(j)}\gamma]$; $j = 1, 2, \ldots, n$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one. The combined invariants of the argument tensors obviously remain the same as for $[{}^{(0)}_{\;d}\bar{\sigma}]$ i.e., ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M$. Then we can express ${}^{(0)}\bar{\boldsymbol{q}}$ as a linear combination of the combined generators $\{{}^{q}\underset{\sim}{G}^i\}$; $i = 1, 2, \ldots, \widetilde{N}$.

$$
{}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{\widetilde{N}} {}^{q}\alpha^i \{{}^{q}\underset{\sim}{G}^i\} \quad (3.13)
$$

The absence of unit vector in (3.13) *as a generator is due to the fact that uniform temperature field does not contribute to* ${}^{(0)}\bar{\boldsymbol{q}}$*. The negative sign in* (3.13) *is because a positive* ${}^{(0)}\bar{\boldsymbol{q}}$ *in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter*. The coefficients ${}^{q}\alpha^i$; $i = 1, 2, \ldots, \widetilde{N}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j; j =$

$1,2,\ldots,M$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1,2,\ldots,M$. To determine the coefficients ${}^q\alpha^i$; $i = 1,2,\ldots,\widetilde{N}$ (in the current configuration at time $t = t_{n+1}$) in (3.13), we consider Taylor series expansion of each ${}^q\alpha^i$; $i = 1,2,\ldots,\widetilde{N}$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1,2,\ldots,M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^q\alpha^i = {}^q\alpha^i\big|_{t_n} + \sum_{j=1}^{M} \frac{\partial({}^q\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \left(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n} \right) + \frac{\partial({}^q\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 1,2,\ldots,\widetilde{N} \quad (3.14) $$

${}^q\alpha^i\big|_{t_n}$, $\frac{\partial({}^q\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1,2,\ldots,M$ and $\frac{\partial({}^q\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1,2,\ldots,\widetilde{N}$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,\ldots,M$ whereas in (3.14), ${}^q\alpha^i = {}^q\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1,2,\ldots,M, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1,2,\ldots,M\big)$; $i = 1,2,\ldots,\widetilde{N}$. When (3.14) is substituted in (3.13), we obtain the final expression for the most general rate constitutive theory of order n for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermoelastic solids. This rate theory uses integrity and hence it is also complete.

3.4.3 Remarks:

1. In sections 3.4.1 - 3.4.2 we have presented n^{th} order rate constitutive theories for the deviatoric Cauchy stress tensor and the heat vector using $[{}^{(1)}_{\;d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ as dependent variables with $[{}^{(j)}\gamma]$; $j = 1,2,\ldots,n$ strain rate tensors as their argument tensors, in addition to $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$. Hence, these developments are independent of the basis.

2. By replacing $[{}^{(1)}_{\;d}\bar{\sigma}]$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]$; $j = 1,2,\ldots,n$ with the appropriate corresponding measures in the chosen basis, we can readily obtain the n^{th} order rate theories for the deviatoric Cauchy stress tensor and heat vector in the desired basis. More specifically we use the following measures:

$$ \begin{array}{lllll} \text{Contravariant basis:} & [{}_d\bar{\sigma}^{(1)}] \;, & \bar{\boldsymbol{q}}^{(0)} \;, & [\gamma^{(j)}] & ; \; j = 1,2,\ldots,n \\ \text{Covariant basis:} & [{}_d\bar{\sigma}_{(1)}] \;, & \bar{\boldsymbol{q}}_{(0)} \;, & [\gamma_{(j)}] & ; \; j = 1,2,\ldots,n \\ \text{Jaumann:} & [{}^{(1)}_{\;d}\bar{\sigma}^J] \;, & {}^{(0)}\bar{\boldsymbol{q}}^J \;, & [{}^{(j)}\gamma^J] & ; \; j = 1,2,\ldots,n \end{array} \quad (3.15) $$

3. Since the tensor $\bar{\boldsymbol{g}}$ is independent of the basis, the combined generators and the combined invariants used in sections 3.4.1 - 3.4.2 only need to be redefined using the convected rates $[\gamma^{(j)}]$; $j = 1, 2, \ldots, n$, $[\gamma_{(j)}]$; $j = 1, 2, \ldots, n$ and $[{}^{(j)}\gamma^J]$; $j = 1, 2, \ldots, n$ for the contravariant, covariant and the Jaumann n^{th} order rate theories.

4. In the final expression for $[{}^{(1)}_{\ d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ containing sum of many group of terms, we consider the following arrangement (in general).

   (a) In each group, *the terms that are defined in the configuration at time* $t = t_n$ *are grouped to define material coefficients*.

   (b) With choice (a), the expression for $[{}^{(1)}_{\ d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ will now consist of the sum of the material coefficients defined in the configuration at $t = t_n$ multiplied with the generators and/or invariants in the current configuration at time $t = t_{n+1}$ in which the deformation is not known.

   (c) The material coefficients defined in (a) will be functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$.

   We follow this arrangement (as far as possible) in all subsequent derivations. *These theories use integrity and hence are complete* but are too complicated and unpractical as they contain too many material coefficients that must be determined experimentally and/or empirically. Details of (a) - (c) are clearly shown in sections 3.7.1 and 3.7.2.

5. Dependence of the coefficients in the final form of the constitutive equations for the deviatoric Cauchy stress tensor and heat vector on $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$ permits variable material coefficients during the evolution. Thus material coefficients can be a function of density and temperature during the evolution for which experimental and/or empirical relations such as power law, the Sutherland law, etc. are justified. Furthermore, dependence of the coefficients on the invariants $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$ permits complex description of material coefficients on the deformation field. Behaviors similar to shear thinning, shear

thickening described by power law, the Carreau-Yasuda model, etc., based on experiments and/or empirical relations for fluids [55] are permissible within the framework of the theory presented here.

6. An important point to note is that *the material coefficients in the final form of the constitutive equations are defined using the configuration at time* $t = t_n$ *whereas the constitutive equations hold for the current configuration at time* $t = t_{n+1}$. This of course is a consequence of the Taylor series expansion of the coefficients in the linear combination using generators about the configuration at time $t = t_n$. In the currently used models in the published works [22, 38, 39] for variable material coefficients, the coefficients are expressed as a function of the unknown deformation field in the current configuration at time $t = t_{n+1}$. This is obviously not supported by the derivations of the constitutive theories presented here in sections 3.4.1 and 3.4.2.

3.5 Rate constitutive theory of order two (*n*=2): compressible thermoelastic solids

If we limit the convected time derivatives of the strain tensor to just first and second as argument tensors in the constitutive theory, i.e., if we only consider $[^{(1)}\gamma]$ and $[^{(2)}\gamma]$ as argument tensors, then we can explicitly present specific forms and expressions for the combined generators and invariants in the constitutive theory. This defines thermoelastic solids of order two.

$$\begin{aligned} [^{(1)}_{\;d}\bar{\sigma}] &= [^{(1)}_{\;d}\bar{\sigma}(\bar{\rho}\ ,\ [^{(1)}\gamma]\ ,\ [^{(2)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \\ {}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [^{(1)}\gamma]\ ,\ [^{(2)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \end{aligned} \tag{3.16}$$

3.5.1 Constitutive theory for the deviatoric Cauchy stress tensor

The combined generators $[^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \ldots, 12$ of the argument tensors $[^{(1)}\gamma]$, $[^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two are listed in table 3.1. The combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}; j =$

$1, 2, \ldots, 16$ of the tensors $[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ are listed in table 3.2 [3–21].

Table 3.1: Combined generators for $[{}^{(1)}_{\,d}\bar{\sigma}]$

	Arguments	**Generators**
(1)	none	$[I]$
(2)	one at a time (including (1))	
	$[{}^{(1)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^{1}] = [{}^{(1)}\gamma]$; $[{}^{\sigma}\underset{\sim}{G}^{2}] = [{}^{(1)}\gamma]^2$
	$[{}^{(2)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^{3}] = [{}^{(2)}\gamma]$; $[{}^{\sigma}\underset{\sim}{G}^{4}] = [{}^{(2)}\gamma]^2$
	$\bar{\boldsymbol{g}}$	$[{}^{\sigma}\underset{\sim}{G}^{5}] = \bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$
(3)	two at a time (including (1) and (2))	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^{6}] = [{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]$
		$[{}^{\sigma}\underset{\sim}{G}^{7}] = [{}^{(1)}\gamma]^2[{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]^2$
		$[{}^{\sigma}\underset{\sim}{G}^{8}] = [{}^{(1)}\gamma][{}^{(2)}\gamma]^2 + [{}^{(2)}\gamma]^2[{}^{(1)}\gamma]$
	$[{}^{(1)}\gamma]$, $\bar{\boldsymbol{g}}$	$[{}^{\sigma}\underset{\sim}{G}^{9}] = \bar{\boldsymbol{g}} \otimes [{}^{(1)}\gamma]\bar{\boldsymbol{g}} + [{}^{(1)}\gamma]\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$
		$[{}^{\sigma}\underset{\sim}{G}^{10}] = \bar{\boldsymbol{g}} \otimes [{}^{(1)}\gamma]^2\bar{\boldsymbol{g}} + [{}^{(1)}\gamma]^2\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$
	$[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	$[{}^{\sigma}\underset{\sim}{G}^{11}] = \bar{\boldsymbol{g}} \otimes [{}^{(2)}\gamma]\bar{\boldsymbol{g}} + [{}^{(2)}\gamma]\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$
		$[{}^{\sigma}\underset{\sim}{G}^{12}] = \bar{\boldsymbol{g}} \otimes [{}^{(2)}\gamma]^2\bar{\boldsymbol{g}} + [{}^{(2)}\gamma]^2\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$

Table 3.2: Combined invariants for $[{}^{(1)}_{\;d}\bar{\sigma}]$: Also valid for ${}^{(0)}\bar{\boldsymbol{q}}$

	Arguments	**Invariants**
(1)	one at a time	
	$[{}^{(1)}\gamma]$	${}^{q\sigma}\underset{\sim}{I}^{1} = \text{tr}([{}^{(1)}\gamma])$; ${}^{q\sigma}\underset{\sim}{I}^{2} = \text{tr}([{}^{(1)}\gamma]^2)$ ${}^{q\sigma}\underset{\sim}{I}^{3} = \text{tr}([{}^{(1)}\gamma]^3)$
	$[{}^{(2)}\gamma]$	${}^{q\sigma}\underset{\sim}{I}^{4} = \text{tr}([{}^{(2)}\gamma])$; ${}^{q\sigma}\underset{\sim}{I}^{5} = \text{tr}([{}^{(2)}\gamma]^2)$ ${}^{q\sigma}\underset{\sim}{I}^{6} = \text{tr}([{}^{(2)}\gamma]^3)$
	$\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{7} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$
(2)	two at a time $\big($ including (1) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$	${}^{q\sigma}\underset{\sim}{I}^{8} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma])$; ${}^{q\sigma}\underset{\sim}{I}^{9} = \text{tr}([{}^{(1)}\gamma]^2[{}^{(2)}\gamma])$ ${}^{q\sigma}\underset{\sim}{I}^{10} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma]^2)$; ${}^{q\sigma}\underset{\sim}{I}^{11} = \text{tr}([{}^{(1)}\gamma]^2[{}^{(2)}\gamma]^2)$ (a) $\boxed{\begin{array}{l} {}^{q\sigma}\underset{\sim}{I} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]) \\ {}^{q\sigma}\underset{\sim}{I} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]) \end{array}}$
	$[{}^{(1)}\gamma]$, $\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{12} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}}$; ${}^{q\sigma}\underset{\sim}{I}^{13} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}}$
	$[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{14} = \bar{\boldsymbol{g}} \cdot [{}^{(2)}\gamma]\,\bar{\boldsymbol{g}}$; ${}^{q\sigma}\underset{\sim}{I}^{15} = \bar{\boldsymbol{g}} \cdot [{}^{(2)}\gamma]^2\,\bar{\boldsymbol{g}}$
(3)	three at a time $\big($ including (1) and (2) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{16} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma][{}^{(2)}\gamma]\,\bar{\boldsymbol{g}}$ (b) $\boxed{\begin{array}{l} {}^{q\sigma}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}} \\ {}^{q\sigma}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}} \end{array}}$

***Remarks*:**

(i) We note that the invariants listed in table 3.2 under (2) (marked (a)) need not be included due to the fact that

$$\text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]) + \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]) = 2\text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma])$$

which is same as ${}^{q\sigma}\underset{\sim}{I}^{8}$ (except the factor 2, which is of no consequence). In many published works (a) are also included in addition to ${}^{q\sigma}\underset{\sim}{I}^{8}$ which is redundant.

(ii) Likewise, in many published works the invariant ${}^{q\sigma}\underset{\sim}{I}^{16}$ is replaced with the two invariants listed under item (3) (marked (b)). Following (i), the sum of the invariants marked (b) is two times ${}^{q\sigma}\underset{\sim}{I}^{16}$. Hence including these in place of ${}^{q\sigma}\underset{\sim}{I}^{16}$ is inappropriate as well.

Now we can express $[{}^{(1)}_{\;d}\bar{\sigma}]$ as a linear combination of $[I]$ and the generators $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \dots, 12$

$$[{}^{(1)}_{\;d}\bar{\sigma}] = {}^{\sigma}\alpha^{0}[I] + \sum_{i=1}^{12} {}^{\sigma}\alpha^{i}[{}^{\sigma}\underset{\sim}{G}^{i}] \tag{3.17}$$

and the coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \dots, 12$ in (3.17) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, 16$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}$; $j = 1, 2, \dots, 16$. To determine the coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \dots, 12$ in (3.17) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \dots, 12$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, 16$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\big|_{t_n} + \sum_{j=1}^{16} \frac{\partial({}^{\sigma}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{i})}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 0, 1, \dots, 12 \tag{3.18}$$

${}^{\sigma}\alpha^{i}\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\big|_{t_n}$; $j = 1, 2, \dots, 16$ and $\frac{\partial({}^{\sigma}\alpha^{i})}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \dots, 12$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}$; $j = 1, 2, \dots, 16$ whereas in (3.18), ${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}; j = 1, 2, \dots, 16, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}; j = 1, 2, \dots, 16\big)$; $i = 0, 1, \dots, 12$. When (3.18) is substituted in (3.17), we obtain the final form of the most general second order ($n = 2$) rate constitutive theory for $[{}^{(1)}_{\;d}\bar{\sigma}]$ for compressible thermoelastic solids. This theory uses integrity and hence is complete. We follow remarks in section 3.4.3 to define material coefficients in the final expression for $[{}^{(1)}_{\;d}\bar{\sigma}]$ and to obtain the corresponding rate theories in contravariant basis, covariant basis and using Jaumann rates.

3.5.2 Constitutive theory for the heat vector

The combined generators $\{{}^{q}\underset{\sim}{G}^{i}\}$; $i = 1, 2, \dots, 7$ of the argument tensors $[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one are given in table 3.3 [3–21]. The combined invariants of the argument tensors obviously remain the same as listen in table 3.2 i.e., ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, 16$. Using the combined generators $\{{}^{q}\underset{\sim}{G}^{i}\}$; $i = 1, 2, \dots, 7$ we can write

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{7} {}^{q}\alpha^{i}\{{}^{q}\underset{\sim}{G}^{i}\} \tag{3.19} $$

The rational for omitting the unit vector and using negative sign in (3.19) has already been explained in section 3.4.2. The coefficients ${}^{q}\alpha^{i}$; $i = 1, 2, \dots, 7$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^{j}; j = 1, 2, \dots, 16$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}$; $j = 1, 2, \dots, 16$. To determine the coefficients ${}^{q}\alpha^{i}$; $i = 1, 2, \dots, 7$ (in the current configuration at time $t = t_{n+1}$) in (3.19), we consider Taylor series expansion of each ${}^{q}\alpha^{i}$; $i = 1, 2, \dots, 7$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, 16$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{q}\alpha^{i} = {}^{q}\alpha^{i}\Big|_{t_n} + \sum_{j=1}^{16} \frac{\partial({}^{q}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\bigg|_{t_n} \Big(({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n} \Big) + \frac{\partial({}^{q}\alpha^{i})}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 1, 2, \dots, 7 \tag{3.20} $$

${}^{q}\alpha^{i}\big|_{t_n}$, $\frac{\partial({}^{q}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\big|_{t_n}$; $j = 1, 2, \dots, 16$ and $\frac{\partial({}^{q}\alpha^{i})}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1, 2, \dots, 7$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}$; $j = 1, 2, \dots, 16$ whereas in (3.20), ${}^{q}\alpha^{i} = {}^{q}\alpha^{i}\big(\bar{\rho}_{t_n}\,,\, \bar{\theta}_{t_n}\,,\, ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}; j = 1, 2, \dots, 16\,,\, \bar{\theta}_{t_{n+1}}\,,\, ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}; j = 1, 2, \dots, 16\big)$; $i = 1, 2, \dots, 7$. When (3.20) is substituted in (3.19), we obtain the final expression for the most general second order ($n = 2$) rate constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermoelastic solids. In this case also, we follow remarks in section 3.4.3 to obtain material coefficients and specific details of the theories in a desired basis.

Table 3.3: Combined generators for ${}^{(0)}\bar{\boldsymbol{q}}$

	Arguments	Generators
(1)	one at a time	
	$[{}^{(1)}\gamma]$	none
	$[{}^{(2)}\gamma]$	none
	$\bar{\boldsymbol{g}}$	$\{{}^{q}\underset{\sim}{G}^{1}\} = \bar{\boldsymbol{g}}$
(2)	two at a time $\big($ including (1) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$	none
	$[{}^{(1)}\gamma]$, $\bar{\boldsymbol{g}}$	$\{{}^{q}\underset{\sim}{G}^{2}\} = [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}}$ $\{{}^{q}\underset{\sim}{G}^{3}\} = [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}}$
	$[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	$\{{}^{q}\underset{\sim}{G}^{4}\} = [{}^{(2)}\gamma]\,\bar{\boldsymbol{g}}$ $\{{}^{q}\underset{\sim}{G}^{5}\} = [{}^{(2)}\gamma]^2\,\bar{\boldsymbol{g}}$
(3)	three at a time $\big($ including (1) and (2) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	$\{{}^{q}\underset{\sim}{G}^{6}\} = \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}}$ $\{{}^{q}\underset{\sim}{G}^{7}\} = \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}}$

3.6 Rate constitutive theory of order one (*n*=1): compressible thermoelastic solids

In this theory we limit the convected time derivative of the strain tensor to just one i.e., we only consider first convected time derivative of the strain tensor as argument of the dependent variables in the constitutive theory

$$\begin{aligned}[{}^{(1)}_{d}\bar{\sigma}] &= [{}^{(1)}_{d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \\ {}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})\end{aligned} \tag{3.21}$$

3.6.1 Constitutive theory for the deviatoric Cauchy stress tensor

The combined generators $[{}^{\sigma}\underset{\sim}{G}^{i}]$ of the argument tensors $[{}^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two (obtained using table 3.1) are given by

$$\begin{gathered}[{}^{\sigma}\underset{\sim}{G}^{1}] = [{}^{(1)}\gamma] \quad ; \quad [{}^{\sigma}\underset{\sim}{G}^{2}] = [{}^{(1)}\gamma]^2 \quad ; \quad [{}^{\sigma}\underset{\sim}{G}^{5}] = \bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}} \\ [{}^{\sigma}\underset{\sim}{G}^{9}] = \bar{\boldsymbol{g}} \otimes [{}^{(1)}\gamma]\bar{\boldsymbol{g}} + [{}^{(1)}\gamma]\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}} \quad ; \quad [{}^{\sigma}\underset{\sim}{G}^{10}] = \bar{\boldsymbol{g}} \otimes [{}^{(1)}\gamma]^2\bar{\boldsymbol{g}} + [{}^{(1)}\gamma]^2\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}\end{gathered} \tag{3.22}$$

Let us redefine $[{}^{\sigma}\underset{\sim}{G}^{5}]$ as $[{}^{\sigma}\underset{\sim}{G}^{3}]$, $[{}^{\sigma}\underset{\sim}{G}^{9}]$ as $[{}^{\sigma}\underset{\sim}{G}^{4}]$ and $[{}^{\sigma}\underset{\sim}{G}^{10}]$ as $[{}^{\sigma}\underset{\sim}{G}^{5}]$. Thus the combined generators are $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \dots, 5$. The combined invariants of the tensors $[{}^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ are (obtained using table 3.2)

$$\begin{gathered}{}^{q\sigma}\underset{\sim}{I}^{1} = \text{tr}([{}^{(1)}\gamma]) \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^{2} = \text{tr}([{}^{(1)}\gamma]^2) \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^{3} = \text{tr}([{}^{(1)}\gamma]^3) \\ {}^{q\sigma}\underset{\sim}{I}^{7} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}} \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^{12} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}} \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^{13} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}}\end{gathered} \tag{3.23}$$

Let us redefine ${}^{q\sigma}\underset{\sim}{I}^{7}$ as ${}^{q\sigma}\underset{\sim}{I}^{4}$, ${}^{q\sigma}\underset{\sim}{I}^{12}$ as ${}^{q\sigma}\underset{\sim}{I}^{5}$ and ${}^{q\sigma}\underset{\sim}{I}^{13}$ as ${}^{q\sigma}\underset{\sim}{I}^{6}$. Thus the combined invariants are ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, 6$. Therefore

$$[{}^{(1)}_{d}\bar{\sigma}] = {}^{\sigma}\alpha^{0}[I] + \sum_{i=1}^{5} {}^{\sigma}\alpha^{i}[{}^{\sigma}\underset{\sim}{G}^{i}] \tag{3.24}$$

The coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \dots, 5$ in (3.24) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, 6$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}$; $j = 1, 2, \dots, 6$. To determine the coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \dots, 5$ in (3.24) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 5$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 6$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{t_n} + \sum_{j=1}^{6} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\Bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ ; \ i = 0, 1, \dots, 5 \tag{3.25} $$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, 6$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \dots, 5$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, 6$ whereas in (3.25), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, 6\ ,\ \bar{\theta}_{t_{n+1}}\ ,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \dots, 6\big)$; $i = 0, 1, \dots, 5$. When (3.25) is substituted in (3.24), we obtain the final expression for the most general first ($n = 1$) order rate constitutive theory for $[{}^{(1)}_{\ d}\bar{\sigma}]$ for compressible thermoelastic solids. This theory uses integrity and hence is complete. We follow section 3.4.3 to obtain material coefficients and the rate constitutive theories in various bases.

3.6.2 Constitutive theory for the heat vector

The combined generators of the argument tensors $[{}^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one (obtained using table 3.3).

$$ \{{}^{q}\underset{\sim}{G}^1\} = \bar{\boldsymbol{g}} \qquad ; \qquad \{{}^{q}\underset{\sim}{G}^2\} = [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}} \qquad ; \qquad \{{}^{q}\underset{\sim}{G}^3\} = [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}} \tag{3.26} $$

The combined invariants remain the same as defined by (3.23). Using the combined generators (3.26) we can write

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{3} {}^{q}\alpha^i \{{}^{q}\underset{\sim}{G}^i\} \tag{3.27} $$

The coefficients ${}^{q}\alpha^i$; $i = 1, 2, 3$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j; j = 1, 2, \dots, 6$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \dots, 6$. To evaluate the coefficients ${}^{q}\alpha^i$; $i = 1, 2, 3$ (in the current configuration at time $t = t_{n+1}$) in (3.27), we consider Taylor series expansion of each ${}^{q}\alpha^i$; $i = 1, 2, 3$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$

; $j = 1, 2, \ldots, 6$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^q\!\alpha^i = {}^q\!\alpha^i\Big|_{t_n} + \sum_{j=1}^{6} \frac{\partial({}^q\!\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\Bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^q\!\alpha^i)}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 1, 2, 3 \tag{3.28}
$$

${}^q\!\alpha^i\big|_{t_n}$, $\frac{\partial({}^q\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \ldots, 6$ and $\frac{\partial({}^q\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1, 2, 3$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, 6$ whereas in (3.28), ${}^q\!\alpha^i = {}^q\!\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 6, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \ldots, 6\big)$; $i = 1, 2, 3$. When (3.28) is substituted in (3.27), we obtain the final expression for the most general first order ($n = 1$) rate constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermoelastic solids. This theory uses integrity and hence is complete. We follow section 3.4.3 to obtain material coefficients and the rate constitutive theories in various bases.

3.7 Constitutive theory for compressible generalized hypo-thermoelastic and hypo-thermoelastic solids

The rate constitutive theory of order one presented in section 3.6 can be modified to obtain the constitutive theory for generalized hypo-thermoelastic and hypo-thermoelastic solids. We consider the first order rate theory presented in section 3.6 and assume that the deviatoric Cauchy stress tensor does not depend upon $\bar{\boldsymbol{g}}$ i.e., $\bar{\boldsymbol{g}}$ is not an argument tensor of $[{}^{(1)}_{\ d}\bar{\sigma}]$, and that the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ does not depend upon $[{}^{(1)}\gamma]$ i.e., $[{}^{(1)}\gamma]$ is not an argument tensor of ${}^{(0)}\bar{\boldsymbol{q}}$.

$$
[{}^{(1)}_{\ d}\bar{\sigma}] = [{}^{(1)}_{\ d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta})] \tag{3.29}
$$

$$
{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}\big) \tag{3.30}
$$

Using (3.29) and (3.30) we can derive a much simplified constitutive theory which obviously has limitations due to limiting the argument tensors in (3.29) and (3.30).

3.7.1 Constitutive theory for the deviatoric Cauchy stress tensor

In this case the generators and invariants are only due to $[^{(1)}\gamma]$. Thus we have

$$[{}^{\sigma}\underset{\sim}{G}^1] = [^{(1)}\gamma] \qquad ; \qquad [{}^{\sigma}\underset{\sim}{G}^2] = [^{(1)}\gamma]^2 \tag{3.31}$$

$${}^{\sigma}\underset{\sim}{I}^1 = \text{tr}([^{(1)}\gamma]) = i_{{}_{(1)}\gamma} \quad ; \quad {}^{\sigma}\underset{\sim}{I}^2 = \text{tr}([^{(1)}\gamma]^2) = ii_{{}_{(1)}\gamma} \quad ; \quad {}^{\sigma}\underset{\sim}{I}^3 = \text{tr}([^{(1)}\gamma]^3) = iii_{{}_{(1)}\gamma} \tag{3.32}$$

$$[^{(1)}_{\,d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{\sigma}\underset{\sim}{G}^1] + {}^{\sigma}\alpha^2[{}^{\sigma}\underset{\sim}{G}^2] \tag{3.33}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0,1,2$ in (3.33) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1,2,3$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1,2,3$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0,1,2$ in (3.33) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0,1,2$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1,2,3$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{3} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 0,1,2 \tag{3.34}$$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1,2,3$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0,1,2$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,3$ whereas in (3.34), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\,,\, \bar{\theta}_{t_n}\,,\, ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1,2,3\,,\, \bar{\theta}_{t_{n+1}}\,,\, ({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1,2,3\big)$. If we let $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)} = {}^{\sigma}\alpha^i{}_{,j}$; $j = 1,2,3$ then (3.34) can be written as

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{3} ({}^{\sigma}\alpha^i{}_{,j})_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 0,1,2 \tag{3.35}$$

Substituting from (3.32) into (3.35) and then from (3.35) into (3.33) gives the following expression

$$
\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] =& \Big({}^{\sigma}\alpha^0\big|_{t_n} + ({}^{\sigma}\alpha^0{}_{,1})_{t_n}\big((i_{{}^{(1)}\gamma})_{t_{n+1}} - (i_{{}^{(1)}\gamma})_{t_n}\big) + ({}^{\sigma}\alpha^0{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
&+ ({}^{\sigma}\alpha^0{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] \quad + \\
&\Big({}^{\sigma}\alpha^1\big|_{t_n} + ({}^{\sigma}\alpha^1{}_{,1})_{t_n}\big((i_{{}^{(1)}\gamma})_{t_{n+1}} - (i_{{}^{(1)}\gamma})_{t_n}\big) + ({}^{\sigma}\alpha^1{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
&+ ({}^{\sigma}\alpha^1{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma] \quad + \\
&\Big({}^{\sigma}\alpha^2\big|_{t_n} + ({}^{\sigma}\alpha^2{}_{,1})_{t_n}\big((i_{{}^{(1)}\gamma})_{t_{n+1}} - (i_{{}^{(1)}\gamma})_{t_n}\big) + ({}^{\sigma}\alpha^2{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
&+ ({}^{\sigma}\alpha^2{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma]^2
\end{aligned}
\tag{3.36}
$$

In (3.36), we note that all quantities at time t_n are known as these correspond to a configuration for which the deformation field is known. We collect terms in (3.36) and define material coefficients and others. Let

$$
\begin{aligned}
&\bar{\sigma}_0\big|_{t_n} = {}^{\sigma}\alpha^0\big|_{t_n} - ({}^{\sigma}\alpha^0{}_{,1})_{t_n}(i_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^0{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^0{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_1 = ({}^{\sigma}\alpha^0{}_{,1})_{t_n} \quad ; \quad {}^{\sigma}b_2 = ({}^{\sigma}\alpha^0{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_3 = ({}^{\sigma}\alpha^0{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^1 = {}^{\sigma}\alpha^1\big|_{t_n} - ({}^{\sigma}\alpha^1{}_{,1})_{t_n}(i_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^1{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^1{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_2^1 = ({}^{\sigma}\alpha^1{}_{,1})_{t_n} \quad ; \quad {}^{\sigma}b_3^1 = ({}^{\sigma}\alpha^1{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_4^1 = ({}^{\sigma}\alpha^1{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^2 = {}^{\sigma}\alpha^2\big|_{t_n} - ({}^{\sigma}\alpha^2{}_{,1})_{t_n}(i_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^2{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^2{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_2^2 = ({}^{\sigma}\alpha^2{}_{,1})_{t_n} \quad ; \quad {}^{\sigma}b_3^2 = ({}^{\sigma}\alpha^2{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_4^2 = ({}^{\sigma}\alpha^2{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^3 = \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^{\sigma}b_2^3 = \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^{\sigma}b_3^3 = \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n}
\end{aligned}
\tag{3.37}
$$

Using (3.37) we can write (3.36) as

$$
\begin{aligned}
[{}^{(1)}_{\;d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] &+ {}^{\sigma}b_1(i_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] \\
&+ {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_2^1(i_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] + {}^{\sigma}b_3^1(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] + {}^{\sigma}b_4^1(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] \\
&+ {}^{\sigma}b_1^2[{}^{(1)}\gamma]^2 + {}^{\sigma}b_2^2(i_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 + {}^{\sigma}b_3^2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 + {}^{\sigma}b_4^2(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 \\
&+ {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] + {}^{\sigma}b_2^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma] + {}^{\sigma}b_3^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma]^2
\end{aligned}
\tag{3.38}
$$

in which $\bar{\sigma}_0\big|_{t_n}$, ${}^{\sigma}b_j$; $j = 1,2,3$, ${}^{\sigma}b_i^1$, ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_k^3$; $k = 1,2,3$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,3$ as evident from (3.37). $\bar{\sigma}_0\big|_{t_n}$ is the initial stress field associated with the configuration at time $t = t_n$. ${}^{\sigma}b_j$; $j = 1,2,3$, ${}^{\sigma}b_i^1$, ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_k^3$; $k = 1,2,3$ are variable material coefficients that are deformation dependent during the evolution. (3.38) are the constitutive equations for $[{}^{(1)}_{\;d}\bar{\sigma}]$ for *compressible generalized hypo-thermoelastic solids* or *compressible hypo-thermoelastic solids*. This theory requires determination of variable material coefficients ${}^{\sigma}b_j$; $j = 1,2,3$, ${}^{\sigma}b_i^1$, ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_k^3$; $k = 1,2,3$, a total of fourteen.

(a) Further assumptions and simplifications

The constitutive theory described by (3.38) can be further simplified if we make the following assumptions:

(i) We neglect the generators $[{}^{(1)}\gamma]^2$ all together in the development of the rate constitutive theory. Thus the terms in (3.38) containing the coefficients ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_3^3$ can be deleted.

(ii) In (3.38), we neglect all product terms in the current configuration $t = t_{n+1}$ i.e., the products of generators $[{}^{(1)}\gamma]$ and its invariants can be deleted from (3.38) including the products of $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ with $[{}^{(1)}\gamma]$.

(iii) When using simplifications (i) and (ii), we ensure that the material coefficients defined in

(3.37) are not affected. With these assumptions, (3.38) reduces to

$$
\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] = {} & \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_1(i_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] \\
& + {}^{\sigma}b_1^1[{}^{(1}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{3.39}
$$

Equations (3.39) is a much simplified constitutive theory for $[{}^{(1)}_{d}\bar{\sigma}]$. It only requires determination of five (${}^{\sigma}b_1$, ${}^{\sigma}b_2$, ${}^{\sigma}b_3$, ${}^{\sigma}b_1^1$, ${}^{\sigma}b_1^3$) material coefficients.

(iv) If we further assume the constitutive theory for $[{}^{(1)}_{d}\bar{\sigma}]$ to be linear in the components of $[{}^{(1)}\gamma]$, then $(ii_{{}^{(1)}\gamma})_{t_{n+1}}$ and $(iii_{{}^{(1)}\gamma})_{t_{n+1}}$ terms in (3.39) can be deleted.

$$
[{}^{(0)}_{d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_1(i_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\tag{3.40}
$$

We redefine the material coefficients to conform to commonly used notations

$$
\begin{aligned}
{}^{\sigma}b_1 &= k\big(\bar{\rho}_{t_n}\,,\ \bar{\theta}_{t_n}\,,\ (i_{{}^{(1)}\gamma})_{t_n}\,,\ (ii_{{}^{(1)}\gamma})_{t_n}\,,\ (iii_{{}^{(1)}\gamma})_{t_n}\big) = k_{t_n} \\
{}^{\sigma}b_1^1 &= 2\mu\big(\bar{\rho}_{t_n}\,,\ \bar{\theta}_{t_n}\,,\ (i_{{}^{(1)}\gamma})_{t_n}\,,\ (ii_{{}^{(1)}\gamma})_{t_n}\,,\ (iii_{{}^{(1)}\gamma})_{t_n}\big) = 2\mu_{t_n} \\
{}^{\sigma}b_1^3 &= -\alpha_{tm}\big(\bar{\rho}_{t_n}\,,\ \bar{\theta}_{t_n}\,,\ (i_{{}^{(1)}\gamma})_{t_n}\,,\ (ii_{{}^{(1)}\gamma})_{t_n}\,,\ (iii_{{}^{(1)}\gamma})_{t_n}\big) = -(\alpha_{tm})_{t_n}
\end{aligned}
\tag{3.41}
$$

Based on the assumptions we can write

$$
[{}^{(1)}_{d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + k_{t_n}\mathrm{tr}([{}^{(1)}\gamma])[I] + 2\mu_{t_n}[{}^{(1)}\gamma] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\tag{3.42}
$$

(3.42) represent the most simplified constitutive theory for the deviatoric Cauchy stress tensor for *compressible generalized hypo-thermoelastic solids* or *compressible hypo-thermoelastic solids with variable material coefficients*. μ and k are *shear modulus* and *bulk modulus* and α_{tm} is *thermal modulus*. This constitutive model for $[{}^{(1)}_{d}\bar{\sigma}]$ only requires determination of three material coefficients.

***Remarks*:**

1. We note that $[{}^{(1)}_{d}\bar{\sigma}]$, $\mathrm{tr}([{}^{(1)}\gamma])$ and $[{}^{(1)}\gamma]$ in (3.42) are defined in the configuration at time $t = t_{n+1}$ but the coefficients k_{t_n}, μ_{t_n} and $(\alpha_{tm})_{t_n}$ are obviously defined in the configuration at time $t = t_n$.

2. μ and k are shear modulus and bulk modulus, and α_{tm} is the thermal modulus.

3. By replacing ($[{}^{(1)}_{d}\bar{\sigma}]$, $\mathrm{tr}([{}^{(1)}\gamma])$ and $[{}^{(1)}\gamma]$) with ($[{}_{d}\bar{\sigma}^{(1)}]$, $\mathrm{tr}([\gamma^{(1)}])$ and $[\gamma^{(1)}]$), ($[{}_{d}\bar{\sigma}_{(1)}]$, $\mathrm{tr}([\gamma_{(1)}])$ and $[\gamma_{(1)}]$) and ($[{}^{(1)}_{d}\bar{\sigma}^{J}]$, $\mathrm{tr}([{}^{(1)}\gamma^{J}])$ and $[{}^{(1)}\gamma^{J}]$) in (3.42), we obtain forms of (3.42) in contravariant basis, covariant basis and Jaumann basis. Keeping in mind that the arguments of μ, k and α_{tm} that are dependent on $[{}^{(1)}\gamma]$ are likewise replaced. In this particular case we note that

$$[{}^{(1)}\gamma] = [\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^{J}] = [\bar{D}] \tag{3.43}$$

Hence, right side of (3.42) remains unaffected by the choice of the basis and we can write

$$[{}^{(1)}_{d}\bar{\sigma}] = \left.\bar{\sigma}_0\right|_{t_n}[I] + k_{t_n}\mathrm{tr}([\bar{D}])[I] + 2\mu_{t_n}[\bar{D}] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{3.44}$$

where $[{}^{(1)}_{d}\bar{\sigma}]$ is the first convected time derivative of the deviatoric Cauchy stress tensor and must be replaced by $[{}_{d}\bar{\sigma}^{(1)}]$, $[{}_{d}\bar{\sigma}_{(1)}]$ or $[{}^{(1)}_{d}\bar{\sigma}^{J}]$ .The coefficients k_{t_n}, μ_{t_n} and $(\alpha_{tm})_{t_n}$ become

$$\begin{aligned} k_{t_n} &= k\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,(i_{\bar{D}})_{t_n}\,,\,(ii_{\bar{D}})_{t_n}\,,\,(iii_{\bar{D}})_{t_n}\big) \\ \mu_{t_n} &= \mu\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,(i_{\bar{D}})_{t_n}\,,\,(ii_{\bar{D}})_{t_n}\,,\,(iii_{\bar{D}})_{t_n}\big) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,(i_{\bar{D}})_{t_n}\,,\,(ii_{\bar{D}})_{t_n}\,,\,(iii_{\bar{D}})_{t_n}\big) \end{aligned} \tag{3.45}$$

(b) Variable material coefficients

From (3.41), we note that μ, k and α_{tm} can be functions of density and temperature during the evolution but their values must be evaluated based on $\bar{\rho}$ and $\bar{\theta}$ in the immediately preceding known configuration ($t = t_n$ in this case). (3.41) permit us to use experimentally and/or empirically determined relations for density and temperature dependent μ, k and α_{tm}. Thus models similar to power law, the Sutherland law etc. for thermofluids or others [22] are valid here as well. This permits us to have variable material coefficients during evolution with the exception that μ, k and α_{tm} must be determined in the configuration at $t = t_n$ whereas $[{}^{(1)}_{\ d}\bar{\sigma}]$ and $[\bar{D}]$ in (3.44) hold for the current configuration at $t = t_{n+1}$. This of course is a consequence of Taylor series expansion of the coefficients about the configuration at time $t = t_n$. From (3.41) we also note that μ, k and α_{tm} can be functions of $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$. This allows us to define μ, k and α_{tm} as functions of the three principal invariants of $[\bar{D}]$ using experimental and/or empirical relations. Thus similar to power law, Carreau-Yasuda, and other models for shear thinning and shear thickening thermofluids [55], the empirical and/or experimental relations for μ, k and α_{tm} dependent on invariants of $[\bar{D}]$ are permissible for thermoelastic solids as well.

Definition: Compressible generalized hypo-thermoelastic solids: From (3.41), we note that μ, k and α_{tm} can be functions of $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$. This allows us to determine μ, k and α_{tm} as functions of $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$ experimentally and/or empirically. When μ, k and α_{tm} show dependence on $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$, $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$ as in (3.41), we refer to such materials as compressible generalized hypo-thermoelastic solids with variable material coefficients.

Definition: Compressible hypo-thermoelastic solids: When μ, k and α_{tm} in (3.44) only show dependence on $\bar{\rho}_{t_n}$ and $\bar{\theta}_{t_n}$, i.e. when

$$\begin{aligned} k_{t_n} &= k\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\big) \\ \mu_{t_n} &= \mu\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\big) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\big) \end{aligned} \tag{3.46}$$

then (3.44) and (3.46) describe compressible hypo-thermoelastic solids with variable material coefficients.

***Remarks*:**

1. The first term in (3.44) is due to the initial stress field in the configuration at time t_n and the last term accounts for the stress field created in the current configuration due to the expansion or contraction compared to the configuration at time $t = t_n$.

2. It is important to emphasize that the constitutive equations such as (3.44) hold for the current configuration at $t = t_{n+1}$. Thus in (3.44) $[{}^{(1)}_{\ d}\bar{\sigma}]$, $[\bar{D}]$ are in the current configuration at $t = t_{n+1}$. However μ_{t_n}, k_{t_n} and $(\alpha_{tm})_{t_n}$ are evaluated based on the known deformation field in the configuration corresponding to $t = t_n$. This is a consequence of the Taylor series expansion about the configuration at time $t = t_n$ of the coefficients ${}^{\sigma}\alpha^i$ in (3.33). In the current published works ([22, 38, 39]), this is not the case, but instead μ, k and α_{tm} are treated as the functions of unknown deformation field in the current configuration at time $t = t_{n+1}$ i.e., instead of μ_{t_n}, k_{t_n} and $(\alpha_{tm})_{t_n}$, these are replaced by

$$\begin{aligned}
\mu_{t_{n+1}} &= \mu(\bar{\rho}_{t_{n+1}}\,,\, \bar{\theta}_{t_{n+1}}\,,\, (i_{\bar{D}})_{t_{n+1}}\,,\, (ii_{\bar{D}})_{t_{n+1}}\,,\, (iii_{\bar{D}})_{t_{n+1}}) = \mu \\
k_{t_{n+1}} &= k(\bar{\rho}_{t_{n+1}}\,,\, \bar{\theta}_{t_{n+1}}\,,\, (i_{\bar{D}})_{t_{n+1}}\,,\, (ii_{\bar{D}})_{t_{n+1}}\,,\, (iii_{\bar{D}})_{t_{n+1}}) = k \qquad (3.47)\\
(\alpha_{tm})_{t_{n+1}} &= \alpha_{tm}(\bar{\rho}_{t_{n+1}}\,,\, \bar{\theta}_{t_{n+1}}\,,\, (i_{\bar{D}})_{t_{n+1}}\,,\, (ii_{\bar{D}})_{t_{n+1}}\,,\, (iii_{\bar{D}})_{t_{n+1}}) = \alpha_{tm}
\end{aligned}$$

In (3.47), we have redefined $\mu_{t_{n+1}}$, $k_{t_{n+1}}$ and $(\alpha_{tm})_{t_{n+1}}$ by μ, k and α_{tm} (their values in the current configuration). With the new definitions of μ, k and α_{tm} in (3.47), (3.44) can be written as (using $\bar{\theta}$ for $\bar{\theta}_{t_{n+1}}$)

$$[{}^{(1)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + k\mathrm{tr}([\bar{D}])[I] + 2\mu[\bar{D}] - \alpha_{tm}(\bar{\theta} - \bar{\theta}_{t_n})[I] \qquad (3.48)$$

With the new definitions in (3.47), the material coefficients are now a function of the unknown deformation field in the current configuration at time $t = t_{n+1}$ as $[{}^{(1)}_{\ d}\bar{\sigma}]$ and $[\bar{D}]$ are. The constitutive equation (3.48) with (3.47) are what is used currently in the published works. When the two configurations at time t_n and t_{n+1} are in close proximity of each other in terms of deformation field, using (3.48) with (3.47) may be justified but it is not supported by the derivation of the constitutive theory presented here.

3.7.2 Constitutive theory for the heat vector

Based on (3.30), in this case the only generator is $\bar{\boldsymbol{g}}$, hence we can write the following for the current configuration ($t = t_{n+1}$):

$$^{(0)}\bar{\boldsymbol{q}} = -{}^{q}\alpha\,\bar{\boldsymbol{g}} \tag{3.49}$$

Also, the only invariant is ${}^{q}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$, hence the coefficient ${}^{q}\alpha$ in (3.49) is a function of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q}\underset{\sim}{I})_{t_{n+1}}$. To determine the coefficient ${}^{q}\alpha$ in (3.49) related to the current configuration at time $t = t_{n+1}$, we consider Taylor series expansion of ${}^{q}\alpha$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}$ and retain only up to linear terms in $\bar{\theta}$ and the invariant ${}^{q}\underset{\sim}{I}$.

$${}^{q}\alpha = {}^{q}\alpha\big|_{t_n} + \left.\frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\right|_{t_n} \big(\,({}^{q}\underset{\sim}{I})_{t_{n+1}} - ({}^{q}\underset{\sim}{I})_{t_n}\big) + \left.\frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\right|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \tag{3.50}$$

${}^{q}\alpha\big|_{t_n}$, $\frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\big|_{t_n}$ and $\frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\big|_{t_n}$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q}\underset{\sim}{I})_{t_n}$ whereas from equation (3.50) we have ${}^{q}\alpha = {}^{q}\alpha\big(\bar{\rho}_{t_n}\,,\ \bar{\theta}_{t_n}\,,\ ({}^{q}\underset{\sim}{I})_{t_n}\,,\ \bar{\theta}_{t_{n+1}}\,,\ ({}^{q}\underset{\sim}{I})_{t_{n+1}}\big)$. Substituting from (3.50) into (3.49)

$$^{(0)}\bar{\boldsymbol{q}} = -\Big({}^{q}\alpha\big|_{t_n} + \left.\frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\right|_{t_n} \big(\,({}^{q}\underset{\sim}{I})_{t_{n+1}} - ({}^{q}\underset{\sim}{I})_{t_n}\big) + \left.\frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\right|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)\,\bar{\boldsymbol{g}} \tag{3.51}$$

or $$^{(0)}\bar{\boldsymbol{q}} = -{}^{q}\alpha\big|_{t_n}\,\bar{\boldsymbol{g}} - \left.\frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\right|_{t_n} \big(\,(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_{n+1}} - (\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_n}\big)\,\bar{\boldsymbol{g}} - \left.\frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\right|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\,\bar{\boldsymbol{g}} \tag{3.52}$$

If there is a uniform temperature change between the configurations at times t_n and t_{n+1}, then $\bar{\boldsymbol{g}} = 0$ and hence $^{(0)}\bar{\boldsymbol{q}}$ must be zero. This condition is satisfied by (3.52). In (3.52), we note that all quantities at time t_n are known as these correspond to a configuration for which the deformation field is known. We collect terms in (3.52) and define material coefficients and others. Let

$$
\begin{aligned}
k_{t_n} &= {}^{q}\!\alpha\big|_{t_n} - \left.\frac{\partial({}^{q}\!\alpha)}{\partial({}^{q}\underset{\sim}{I})}\right|_{t_n} (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n} \\
(k_1)_{t_n} &= \left.\frac{\partial({}^{q}\!\alpha)}{\partial({}^{q}\underset{\sim}{I})}\right|_{t_n} \\
(k_2)_{t_n} &= \left.\frac{\partial({}^{q}\!\alpha)}{\partial\bar{\theta}}\right|_{t_n}
\end{aligned}
\tag{3.53}
$$

then (3.52) becomes (we drop the subscript t_{n+1} since it is understood it represents the current configuration)

$$
{}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} - (k_1)_{t_n}(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})\,\bar{\boldsymbol{g}} - (k_2)_{t_n}(\bar{\theta}-\bar{\theta}_{t_n})\,\bar{\boldsymbol{g}} \tag{3.54}
$$

We note that $k_{t_n} = k(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$, $(k_1)_{t_n} = k_1(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$ and $(k_2)_{t_n} = k_2(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$. (3.54) is the most general form of the constitutive equation for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ based on (3.30). If we neglect the last term in (3.54) then

$$
{}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} - (k_1)_{t_n}(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})\,\bar{\boldsymbol{g}} \tag{3.55}
$$

If we neglect infinitesimals of order two and higher in the components of $\bar{\boldsymbol{g}}$, then

$$
{}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} \tag{3.56}
$$

$$
\text{or} \qquad {}^{(0)}\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} = -k[I]\,\bar{\boldsymbol{g}} = -[K]\,\bar{\boldsymbol{g}} \tag{3.57}
$$

in which k is *thermal conductivity* and $[K]$ is the *diagonal thermal conductivity matrix*. We note

$$
k = k_{t_n} = k_{t_n}\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}\big) \tag{3.58}
$$

Based on (3.58), the thermal conductivity can be a function of density, temperature and the first invariant of $\bar{\boldsymbol{g}}$ i.e., $\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}}$. Thus, as in the case of the shear modulus, here also we can use experimental and/or empirical relation for thermal conductivity a a function of $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}}$, keeping in mind that

(3.57) holds for the current configuration at time $t = t_{n+1}$ whereas k in (3.57) is only defined for the configuration at time $t = t_n$. Thus, for example, power law, the Sutherland law for temperature dependent k are justified.

$$k_{t_n} = k^0 \left(\frac{\bar{\theta}_{t_n}}{\theta^0} \right)^{\underline{n}} \quad ; \quad \text{Power law} \tag{3.59}$$

$$k_{t_n} = k^0 \left(\frac{\bar{\theta}_{t_n}}{\theta^0} \right)^{3/2} \left(\frac{\theta^0 + \underline{s}}{\bar{\theta}_{t_n} + \underline{s}} \right) \quad ; \quad \text{Sutherland law} \tag{3.60}$$

k^0, θ^0, $\underline{n}$ and $\underline{s}$ are constants for a specific solid. This permits us to have variable thermal conductivity during the evolution. Similarly we can also consider k as a function of $\bar{\rho}_{t_n}$ as well, keeping in mind that based on (3.58), dependence of k_{t_n} on $(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_n}$ is permissible as well.

***Remarks*:**

1. The constitutive theory for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ based on the generator of the argument tensor $\bar{\boldsymbol{g}}$ given by (3.54) is much more complicated than the standard Fourier heat conduction law defined by (3.57) which only considers first order terms in the components of $\bar{\boldsymbol{g}}$ in the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$.

2. In (3.54) as well as (3.57), $\bar{\boldsymbol{g}}$ is independent of the basis, hence (3.54) and (3.57) are valid in contra- and co-variant basis as well as the Jaumann basis, i.e.

$$ {}^{(0)}\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}_{(0)} = \bar{\boldsymbol{q}}^{J} = \bar{\boldsymbol{q}} \tag{3.61}$$

3. The constitutive equations (3.54) and (3.56) hold for the current configuration at time $t = t_{n+1}$, thus ${}^{(0)}\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ correspond to time $t = t_{n+1}$, however, the coefficients k, ${}^1k$ and ${}^2k$ are defined in the configuration at time $t = t_n$. This is obviously a consequence of the Taylor series expansion of ${}^{\sigma}\alpha$ about the configuration at $t = t_n$. In the currently published works this is not the case, but instead k, ${}^1k$ and ${}^2k$ are treated as functions of the unknown field in

the current configuration, that is k_{t_n}, ${}^1k_{t_n}$ and ${}^2k_{t_n}$ are replaced by

$$k_{t_{n+1}} = k\big(\bar{\rho}_{t_{n+1}}\ ,\ \bar{\theta}_{t_{n+1}}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}}\big) = k \tag{3.62}$$

$${}^1k_{t_{n+1}} = {}^1k\big(\bar{\rho}_{t_{n+1}}\ ,\ \bar{\theta}_{t_{n+1}}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}}\big) = {}^1k \tag{3.63}$$

$${}^2k_{t_{n+1}} = {}^2k\big(\bar{\rho}_{t_{n+1}}\ ,\ \bar{\theta}_{t_{n+1}}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}}\big) = {}^2k \tag{3.64}$$

In (3.62) - (3.64) we redefined $k_{t_{n+1}}$, ${}^1k_{t_{n+1}}$, ${}^2k_{t_{n+1}}$ by k, ${}^1k$, ${}^2k$ in the current configuration at time $t = t_{n+1}$. With these new definitions (3.54) and (3.56) can be written as

$${}^{(0)}\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} - {}^1k\big(\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}} - (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}\big)\,\bar{\boldsymbol{g}} \tag{3.65}$$

$${}^{(0)}\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} \tag{3.66}$$

The power law and the Sutherland law ((3.59) and (3.60)) are accordingly modified

$$k = k^0\Big(\frac{\bar{\theta}}{\theta^0}\Big)^{\underline{n}} \qquad ; \qquad \text{Power law} \tag{3.67}$$

$$k = k^0\Big(\frac{\bar{\theta}}{\theta^0}\Big)^{3/2}\Big(\frac{\theta^0 + \underline{s}}{\bar{\theta} + \underline{s}}\Big) \qquad ; \qquad \text{Sutherland law} \tag{3.68}$$

Equations (3.66) - (3.68) are what is used in the published works. When the two configurations at times t_n and t_{n+1} are in close proximity of each other in terms of the deformation field, using (3.62) - (3.66) may be justified, but it is not supported by the derivation of the constitutive theory presented here.

3.8 Incompressible ordered thermoelastic solids: of orders *n*, 2 and 1

All derivations presented so far for the constitutive theories of orders n, 2 and 1 for $[{}^{(1)}_{\ d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ assumed the thermoelastic solid to be compressible. In this section we consider ordered rate constitutive theories for $[{}^{(1)}_{\ d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ for incompressible thermoelastic solids. In case of incompressible matter

$$\bar{\rho} = \rho_{0} = \text{constant} \tag{3.69}$$

$$\text{div}(\bar{\boldsymbol{v}}) = 0 \tag{3.70}$$

$$\therefore \quad \text{tr}([{}^{(1)}\gamma]) = \text{tr}([\gamma^{(1)}]) = \text{tr}([\gamma_{(1)}]) = \text{tr}([{}^{(1)}\gamma^{J}]) = \text{tr}([\bar{D}]) = 0 \tag{3.71}$$

$$\det([J]) = 1 \tag{3.72}$$

Thus for this case, density $\bar{\rho}$ can be eliminated from the argument tensors of the dependent variables $[{}^{(1)}_{\ d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ in the rate constitutive theory for incompressible thermoelastic solids. Thus, for incompressible thermoelastic solids

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\theta}(\bar{\boldsymbol{x}}, t)\big) \tag{3.73}$$

$$[{}^{(0)}\bar{\sigma}\,] = \big[{}^{(0)}_{\ e}\bar{\sigma}\big(\bar{\theta}(\bar{\boldsymbol{x}}, t)\big)\big] + [{}^{(0)}_{\ d}\bar{\sigma}\,] \tag{3.74}$$

$$[{}^{(1)}_{\ d}\bar{\sigma}\,] = \big[{}^{(1)}_{\ d}\bar{\sigma}\big([{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t)\big)\big] \tag{3.75}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)]\ ;\ j = 1, 2 \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t)\big) \tag{3.76}$$

$$[{}^{(0)}_{\ e}\bar{\sigma}] = \bar{p}(\bar{\theta}(\bar{\boldsymbol{x}}, t))[I] \tag{3.77}$$

The rate constitutive theories of orders n, 2 and 1 for compressible thermoelastic solids presented in earlier sections can be modified by: (i) eliminating $\bar{\rho}$ all together from the entire derivations and (ii) incorporating incompressibility conditions using (3.70) - (3.72) to obtain rate constitutive theories of orders n, 2 and 1 for incompressible thermoelastic solids. Details are straight forward, hence not presented here for the sake of brevity. By replacing $[{}^{(1)}_{\ d}\bar{\sigma}]$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ with the appropriate corresponding measures in the chosen basis, ($[{}_{d}\bar{\sigma}^{(1)}]$, $\bar{\boldsymbol{q}}^{(0)}$, $[\gamma^{(j)}]$; $j = 1, 2, \dots, n$), ($[{}_{d}\bar{\sigma}_{(1)}]$, $\bar{\boldsymbol{q}}_{(0)}$, $[\gamma_{(j)}]$; $j = 1, 2, \dots, n$) and ($[{}^{(1)}{}_{d}\bar{\sigma}^{J}]$, ${}^{(0)}\bar{\boldsymbol{q}}^{J}$, $[{}^{(j)}\gamma^{J}]$; $j = 1, 2, \dots, n$) we can easily obtain the rate theories of orders n, 2 and 1 in contravariant basis, covariant basis and Jaumann basis.

3.9 Constitutive theories for incompressible generalized hypo-thermoelastic and hypo-thermoelastic solids

Following the rate constitutive theories for compressible generalized hypo-thermoelastic and hypo-thermoelastic solids presented in section 3.7 and eliminating $\bar{\rho}$ from the argument tensors, (3.29) - (3.30) reduce to

$$[{}^{(1)}_{\ d}\bar{\sigma}] = [{}^{(1)}_{\ d}\bar{\sigma}([{}^{(1)}\gamma]\ ,\ \bar{\theta})] \tag{3.78}$$

$$^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{3.79}$$

3.9.1 Constitutive theory for the deviatoric Cauchy stress tensor

Generators $[{}^{\sigma}\underset{\sim}{G}^1] = [{}^{(1)}\gamma]$ and $[{}^{\sigma}\underset{\sim}{G}^2] = [{}^{(1)}\gamma]^2$ of the argument tensor $[{}^{(1)}\gamma]$ allow us to write

$$[{}^{(1)}_{\ d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{(1)}\gamma] + {}^{\sigma}\alpha^2[{}^{(1)}\gamma]^2 \tag{3.80}$$

and the invariants of $[{}^{(1)}\gamma]$ are

$$^{\sigma}\underset{\sim}{I}^1 = i_{{}^{(1)}\gamma} = \text{tr}([{}^{(1)}\gamma]) = 0 \quad ; \quad {}^{\sigma}\underset{\sim}{I}^2 = ii_{{}^{(1)}\gamma} = \text{tr}([{}^{(1)}\gamma]^2) \quad ; \quad {}^{\sigma}\underset{\sim}{I}^3 = iii_{{}^{(1)}\gamma} = \text{tr}([{}^{(1)}\gamma]^3) \tag{3.81}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ in (3.80) are functions of temperature $\bar{\theta}$ and the invariants ${}^{\sigma}\underset{\sim}{I}^2$ and ${}^{\sigma}\underset{\sim}{I}^3$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\theta}_{t_{n+1}}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 2, 3$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ in (3.80) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 2, 3$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$\begin{aligned} {}^{\sigma}\alpha^i = \ & {}^{\sigma}\alpha^i\big|_{t_n} + \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^2)}\bigg|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^2)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^2)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^3)}\bigg|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^3)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^3)_{t_n}\big) \\ & + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 0, 1, 2 \end{aligned} \tag{3.82}$$

Let us introduce the notation $\frac{\partial({}^\sigma\alpha^i)}{\partial({}^\sigma \underset{\sim}{I}^j)} = {}^\sigma\alpha^i{}_{,j}$; $j = 2, 3$ and $i = 0, 1, 2$. Substituting from (3.81) into (3.82) and then from (3.82) into (3.80)

$$
\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] = & \Big({}^\sigma\alpha^0\big|_{t_n} + ({}^\sigma\alpha^0{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
& + ({}^\sigma\alpha^0{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^\sigma\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] \quad + \\
& \Big({}^\sigma\alpha^1\big|_{t_n} + ({}^\sigma\alpha^1{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
& + ({}^\sigma\alpha^1{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^\sigma\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma] \quad + \\
& \Big({}^\sigma\alpha^2\big|_{t_n} + ({}^\sigma\alpha^2{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
& + ({}^\sigma\alpha^2{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^\sigma\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma]^2
\end{aligned}
\tag{3.83}
$$

In (3.83), we note that all quantities at time t_n are known as they correspond to a configuration (t_n) for which the deformation field is known. We collect terms in (3.83) and define material coefficients and others. Let

$$
\begin{aligned}
& \bar{\sigma}_0\big|_{t_n} = {}^\sigma\alpha^0\big|_{t_n} - ({}^\sigma\alpha^0{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^\sigma\alpha^0{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
& {}^\sigma b_2 = ({}^\sigma\alpha^0{}_{,2})_{t_n} \quad ; \quad {}^\sigma b_3 = ({}^\sigma\alpha^0{}_{,3})_{t_n} \\
& {}^\sigma b_1^1 = {}^\sigma\alpha^1\big|_{t_n} - ({}^\sigma\alpha^1{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^\sigma\alpha^1{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
& {}^\sigma b_3^1 = ({}^\sigma\alpha^1{}_{,2})_{t_n} \quad ; \quad {}^\sigma b_4^1 = ({}^\sigma\alpha^1{}_{,3})_{t_n} \quad ; \quad {}^\sigma b_3^2 = ({}^\sigma\alpha^2{}_{,2})_{t_n} \quad ; \quad {}^\sigma b_4^2 = ({}^\sigma\alpha^2{}_{,3})_{t_n} \\
& {}^\sigma b_1^2 = {}^\sigma\alpha^2\big|_{t_n} - ({}^\sigma\alpha^2{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^\sigma\alpha^2{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
& {}^\sigma b_1^3 = \frac{\partial({}^\sigma\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^\sigma b_2^3 = \frac{\partial({}^\sigma\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^\sigma b_3^3 = \frac{\partial({}^\sigma\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n}
\end{aligned}
\tag{3.84}
$$

Using (3.84) we can write (3.83) as

$$
\begin{aligned}
[{}^{(1)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] &+ {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] \\
&+ {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_3^1(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] + {}^{\sigma}b_4^1(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] \\
&+ {}^{\sigma}b_1^2[{}^{(1)}\gamma]^2 + {}^{\sigma}b_3^2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 + {}^{\sigma}b_4^2(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 \\
&+ {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] + {}^{\sigma}b_2^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma] + {}^{\sigma}b_3^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma]^2
\end{aligned}
\tag{3.85}
$$

in which the coefficients ${}^{\sigma}b_2$, ${}^{\sigma}b_3$, ${}^{\sigma}b_1^1$, ${}^{\sigma}b_3^1$, ${}^{\sigma}b_4^1$, ${}^{\sigma}b_1^2$, ${}^{\sigma}b_3^2$, ${}^{\sigma}b_4^2$ and ${}^{\sigma}b_1^3$, ${}^{\sigma}b_2^3$, ${}^{\sigma}b_3^3$ are functions of $\bar{\theta}_{t_n}$, $(ii_{{}^{(1)}\gamma})_{t_n}$ and $(iii_{{}^{(1)}\gamma})_{t_n}$ as evident from (3.84). These are the variable material coefficients that are dependent of the deformation during the evolution. $\bar{\sigma}_0\big|_{t_n}$ is the initial stress field associated with the configuration at time $t = t_n$. (3.85) are the constitutive equations for $[{}^{(1)}_{\ d}\bar{\sigma}]$ for *incompressible generalized hypo-thermoelastic solids* or *incompressible hypo-thermoelastic solids with variable material coefficients*. This theory requires determination of a total of eleven variable material coefficients.

(a) Further assumptions and simplifications

The constitutive theory described by (3.85) can be further simplified if we make the following assumptions:

(i) We neglect the generators $[{}^{(1)}\gamma]^2$ all together in the development of the rate theory. Thus the terms in (3.85) containing the coefficients ${}^{\sigma}b_1^2$, ${}^{\sigma}b_3^2$, ${}^{\sigma}b_4^2$ and ${}^{\sigma}b_3^3$ are deleted.

(ii) We also neglect all product terms in the current configuration at $t = t_{n+1}$ i.e., the products of generators $[{}^{(1)}\gamma]$ and its invariants can be deleted from (3.85) including the products of $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ with $[{}^{(1)}\gamma]$.

(iii) When using simplifications (i) and (ii), we ensure that the material coefficients defined in (3.84) are not affected. With these assumptions, (3.85) reduces to

$$
[{}^{(1)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{3.86}
$$

(3.86) is a much simplified constitutive theory for $[{}^{(1)}_{\;d}\bar{\sigma}]$. It only requires determination of four material coefficients.

(iv) If we further assume that the constitutive theory for $[{}^{(0)}_{\;d}\bar{\sigma}]$ to be linear in the components of $[{}^{(1)}\gamma]$, then $(ii_{{}^{(1)}\gamma})_{t_{n+1}}$ and $(iii_{{}^{(1)}\gamma})_{t_{n+1}}$ terms in (3.86) can be deleted.

$$[{}^{(1)}_{\;d}\bar{\sigma}] = \bar{\sigma}_0\Big|_{t_n}[I] + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{3.87}$$

We rename the material coefficients to conform to commonly used notations

$$\begin{aligned} {}^{\sigma}b_1^1 &= 2\mu\big(\bar{\theta}_{t_n}\,,\ (ii_{{}^{(1)}\gamma})_{t_n}\,,\ (iii_{{}^{(1)}\gamma})_{t_n}\big) = 2\mu_{t_n} \\ {}^{\sigma}b_1^3 &= -\alpha_{tm}\big(\bar{\theta}_{t_n}\,,\ (ii_{{}^{(1)}\gamma})_{t_n}\,,\ (iii_{{}^{(1)}\gamma})_{t_n}\big) = -(\alpha_{tm})_{t_n} \end{aligned} \tag{3.88}$$

then we can write

$$[{}^{(1)}_{\;d}\bar{\sigma}] = \ \bar{\sigma}_0\Big|_{t_n}[I] + 2\mu_{t_n}[{}^{(1)}\gamma] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{3.89}$$

Equations (3.89) is the most simplified constitutive theory for $[{}^{(0)}_{\;d}\bar{\sigma}]$ for *incompressible generalized hypo-thermoelastic solids* or *incompressible hypo-thermoelastic solids with variable material coefficients*. μ is the *shear modulus* and α_{tm} is the *thermal modulus*. This theory requires determination of only two material coefficients.

***Remarks*:**

1. We note that (3.89) holds for the deformed configuration at time $t = t_{n+1}$ i.e., current configuration, thus $[{}^{(0)}_{\;d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ are defined at time $t = t_{n+1}$ but the coefficients η_{t_n} and $(\alpha_{tm})_{t_n}$ are obviously defined in the configuration at time $t = t_n$.

2. By replacing $[{}^{(1)}_{\;d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ with the appropriate corresponding measures in the chosen basis, $\big([{}_d\bar{\sigma}^{(1)}], [\gamma^{(1)}]\big)$, $\big([{}_d\bar{\sigma}_{(1)}], [\gamma_{(1)}]\big)$ and $\big([{}^{(1)}_{\;d}\bar{\sigma}^J], [{}^{(1)}\gamma^J]\big)$ we can obtain forms of (3.89) in

contravariant basis, covariant basis and Jaumann basis, keeping in mind that the arguments of μ and α_{tm} dependent on $[{}^{(1)}\gamma]$ are likewise replaced. In (3.89), $ii_{{}^{(1)}\gamma}$ and $iii_{{}^{(1)}\gamma}$ need changes in μ_{t_n} and $(\alpha_{tm})_{t_n}$ due to change of basis.

3. As in case of compressible thermofluids, here also we note that

$$[{}^{(1)}\gamma] = [\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}] \tag{3.90}$$

Here, the right side of (3.89) remains unaffected due to the change of basis and we can write

$$[{}^{(1)}_{\;d}\bar{\sigma}] = \; \bar{\sigma}_0\Big|_{t_n}[I] + 2\mu_{t_n}[\bar{D}] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{3.91}$$

where $[{}^{(1)}_{\;d}\bar{\sigma}]$ is the first convected time derivative of the deviatoric Cauchy stress tensor for incompressible case and the coefficients μ_{t_n} and $(\alpha_{tm})_{t_n}$ become

$$\begin{aligned} \mu_{t_n} \quad &= \mu\big(\bar{\theta}_{t_n} \,,\, (ii_{\bar{D}})_{t_n} \,,\, (iii_{\bar{D}})_{t_n}\big) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}\big(\bar{\theta}_{t_n} \,,\, (ii_{\bar{D}})_{t_n} \,,\, (iii_{\bar{D}})_{t_n}\big) \end{aligned} \tag{3.92}$$

(b) Variable material coefficients

From (3.92), we note that μ and α_{tm} can be functions of temperature during the evolution but their values must be evaluated based on $\bar{\theta}$ in the immediately preceding known configuration ($t = t_n$ in this case). (3.92) permit us to use experimentally and/or empirically determined relations for temperature dependent μ and α_{tm}. For example in (3.91) and (3.92), power law and Sutherland models for $\mu_{t_n} = \mu_{t_n}(\bar{\theta}_{t_n})$ remain valid for the incompressible thermoelastic solids as well.

Definition: Incompressible generalized hypo-thermoelastic solids and hypo-thermoelastic solids: From (3.92), we note that shear modulus μ can be a function of the principal invariants of $[\bar{D}]$ i.e., $ii_{\bar{D}}$ and $iii_{\bar{D}}$. This allows us to express μ a a function of $ii_{\bar{D}}$ and $iii_{\bar{D}}$ using experimental and/or empirical relations between μ and $ii_{\bar{D}}$ and $iii_{\bar{D}}$.

***Remarks*:**

1. When μ and α_{tm} in (3.91) show dependence on $\bar{\theta}_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$, we refer to the solid described by (3.91) as incompressible generalized hypo-thermoelastic solid with variable material properties.

2. When μ and α_{tm} only show dependence on $\bar{\theta}_{t_n}$ i.e., when

$$\begin{aligned} \mu_{t_n} &= \mu(\bar{\theta}_{t_n}) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}(\bar{\theta}_{t_n}) \end{aligned} \tag{3.93}$$

Then, (3.91) and (3.93) describe incompressible hypo-thermoelastic solids with variable material coefficients.

3. Constitutive relations (3.91) hold for the current configuration at time $t = t_{n+1}$. In (3.91) $[{}^{(1)}_{\ d}\bar{\sigma}]$ and $[\bar{D}]$ are in the current configuration at time $t = t_{n+1}$, however μ_{t_n} and $(\alpha_{tm})_{t_n}$ are evaluated based on the known deformation field in the configuration corresponding to $t = t_n$. This is a consequence of the Taylor series expansion about the configuration at time $t = t_n$ of the coefficients ${}^{\sigma}\alpha^i$ used in the linear combination of the generators to define $[{}^{(1)}_{\ d}\bar{\sigma}]$. We remark that in the published works, this is not the case, but instead μ and α_{tm} are treated as the functions of unknown deformation field in the current configuration at time $t = t_{n+1}$ i.e., instead of μ_{t_n} and $(\alpha_{tm})_{t_n}$, these are replaced by

$$\begin{aligned} \mu_{t_{n+1}} &= \mu(\bar{\theta}_{t_{n+1}}\ ,\ (ii_{\bar{D}})_{t_{n+1}}) = \mu \\ (\alpha_{tm})_{t_{n+1}} &= \alpha_{tm}(\bar{\theta}_{t_{n+1}}\ ,\ (ii_{\bar{D}})_{t_{n+1}}) = \alpha_{tm} \end{aligned} \tag{3.94}$$

with these new definitions of μ and α_{tm}, (3.91) can be written as (no need for subscript t_{n+1})

$$[{}^{(1)}_{\ d}\bar{\sigma}] = \ \bar{\sigma}_0\Big|_{t_n}[I] + 2\mu[\bar{D}] - \alpha_{tm}(\bar{\theta} - \bar{\theta}_{t_n})[I] \tag{3.95}$$

$[{}^{(1)}_{\ d}\bar{\sigma}]$, $[\bar{D}]$, μ and α_{tm} are in the current configuration at time $t = t_{n+1}$. With (3.94), variable material coefficients and dependence of μ and α_{tm} are expressed using unknown deformation field in the current configuration at time $t = t_{n+1}$. (3.95) and (3.94) are what is used currently in the published works. When the two configurations at times $t = t_n$ and $t = t_{n+1}$ are in close proximity of each other in terms of deformation field, using (3.95) with (3.94) may be justified but it is not supported by the derivation of the constitutive theory presented here.

3.9.2 Constitutive theory for the heat vector

Based on (3.30), it is obvious that the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ for the incompressible case remains the same as for the compressible case (section 3.7.2) except that $\bar{\rho}$ drops out in the entire derivation as it is not an argument tensor in (3.79). Details can be readily obtained from the derivation in section 3.7.2 and are not repeated here for the sake of brevity.

3.10 Conjugate stress-strain measures and validity of rate constitutive theories in different bases

In the rate constitutive theories presented in this chapter we have considered deviatoric contravariant Cauchy stress tensor and Almansi strain tensor in contravariant basis and their convected time derivatives in the contravariant basis as conjugate measures. Likewise, deviatoric covariant Cauchy stress tensor and Green's strain tensor in covariant basis and their convected time derivatives in the covariant basis are used as conjugate measures in the covariant basis. Various aspects of the conjugate stress and strain measures are considered in references [52–54]. It is now well established [3, 4, 55, 56] that the choice of stress and strain measures must have a common basis and be conjugate in the sense of energy. Based on this, the choices of stress and strain measures and their convected time derivatives considered in the present work are consistent.

Based on thermodynamic considerations i.e., the conditions resulting from the entropy inequality require the work expanded due to the deviatoric Cauchy stress tensor to be positive. The constitutive theory for the deviatoric Cauchy stress tensor only needs to satisfy this condition for thermodynamic equilibrium. The constitutive inequalities presented in references [52–54] are believed to be supported by observed experimental behaviors and applications but do not have thermodynamic basis as stated by the author in reference [52]. The constitutive theories presented in this chapter for finite deformation utilize contravariant, covariant and Jaumann bases. The validity of one constitutive theory over the others can be easily established by going back to the basic definitions of the stress measures. In the definition of the contravariant deviatoric Cauchy stress tensor we utilize actual deformed tetrahedron in the current configuration and the dyads in the contravariant basis based on contravariant base vectors that are perpendicular to the faces of the deformed tetrahedron but form non-orthogonal basis. Upon substituting the expressions for contravariant base vectors (with basis in x-frame) we obtain contravariant deviatoric Cauchy stress tensor in x-frame. In the definition of the covariant deviatoric Cauchy stress tensor we require a new configuration of the deformed tetrahedron such that covariant base vectors are normal to its faces. This requires that the true deformed tetrahedron in the current configuration be further deformed [55, 56] so that covariant base vectors become perpendicular to this new configuration of the tetrahedron. This of course is non-physical i.e., not in agreement with the physics of deformation. In case of finite deformation, the differences between the configurations of actual deformed tetrahedron used in contravariant description and the tetrahedron required for covariant description may be significant. The contravariant description is in agreement with the kinematic of deformation whereas covariant description using non-physical configuration of the tetrahedron in the current configuration is not. With progressively increasing deformation, the contravariant descriptions remain physical and valid whereas the covariant descriptions will become progressively more spurious as the tetrahedron configuration used in their description begins to deviate from the true deformed tetrahedron used in the contravariant descriptions. It is clear that for the same material coefficients, the contra- and co- descriptions will produce different deformation behaviors when the deformation is finite. Only in case of infinitesimal deformation, the two descriptions will yield the same measures of stresses. Jaumann descriptions obviously suffer from the same problem as covariant descriptions, as they are average of co- and contra- descriptions. Thus, the constitutive theories in contra- and co- bases will yield different responses for finite deformation. This issue of different deformation response using contra- and co-variant descriptions for the same material coefficients has also be discussed in references [58, 59]. In reference [30], model problems and numerical studies have been presented to demonstrate and illustrate these various aspects discussed here.

For the deforming matter to be in thermodynamic equilibrium, we only need to satisfy the conditions resulting from the entropy inequality, which requires that the work expanded due to deviatoric Cauchy stress tensor must be positive. As pointed out in references [52–54], there are other constitutive inequalities associated with the stress and strain rates that may be in agreement with what is observed experimentally or in applications but these may not have thermodynamic basis [52]. For the rate constitutive theories presented in this chapter, these aspects need to be further explored to possibly establish new constitutive inequalities that validate the rate theories. This work is currently in progress.

3.11 Numerical studies using linear and non-linear heat conduction laws

In this section, we study the influence of non-linearity in terms of temperature gradient in the constitutive equation for the heat vector, and present comparisons with the commonly used Fourier heat conduction law. We consider heat transfer in an incompressible thermoelastic solid under small motion and infinitesimal deformation, hence the distinction between Eulerian and Lagrangian description disappears and we drop over bar $(\bar{\ })$ on all quantities replacing them with hat $(\hat{\ })$ to emphasize that these quantities have dimensions. Quantities without hat are dimensionless. In addition, the distinction between covariant and contravariant bases also disappears for infinitesimal deformation. In this case, continuity and momentum equations are satisfied identically, hence the mathematical model only consists of energy equation and the non-linear constitutive theory for the heat vector. We consider aluminum [60] with the following material coefficients (assumed constant during evolution) corresponding to 300 Kelvin (26.85 °C)

$$\hat{\rho} = 2700 \text{ kg/m}^3 \qquad ; \qquad \hat{c}_v = 903.0 \text{ J/kg.K} \qquad ; \qquad \hat{k} = 237 \text{ Watt/m.K} \tag{3.96}$$

in which $\hat{\rho}$, $\hat{c}_v$ and $\hat{k}$ are density, specific heat and thermal conductivity. We consider a non-linear constitutive equation for the heat vector (3.55). The mathematical model consists of energy equation and the non-linear constitutive theory for the heat vector and are given in the following

$$\hat{\rho}\, \hat{c}_v \frac{\partial \hat{T}}{\partial \hat{t}} + \text{div}\, \hat{\boldsymbol{q}} = 0 \tag{3.97}$$

$$\hat{\boldsymbol{q}} = -\hat{k}\, \hat{\boldsymbol{g}} - \hat{k}_1\, (\hat{\boldsymbol{g}} \cdot \hat{\boldsymbol{g}})\hat{\boldsymbol{g}} \tag{3.98}$$

where $\hat{k}_1$ is the new coefficient associated with the non-linear term and has units of Watt/m.k^3. We consider an aluminum bar with of length $\hat{L} = 0.1$ m and uniform initial temperature of $\hat{T}_1 = 300$ K. We begin with all quantities with their usual dimensions (units) and then non-dimensionalize them using the following. The quantities with the subscript zero are the reference quantities.

$$\begin{gathered} x = \frac{\hat{x}}{L_0} \quad , \quad T = \frac{\hat{T}}{T_0} \quad , \quad \rho = \frac{\hat{\rho}}{\rho_0} \quad , \quad c_v = \frac{\hat{c}_v}{c_{v_0}} \quad , \quad k = \frac{\hat{k}}{k_0} \quad , \quad t = \frac{\hat{t}}{t_0} \quad , \quad k_1 = \frac{\hat{k}_1}{k_{1_0}} \\ t_0 = \frac{L_0^2\, \rho_0 c_{v_0}}{k_0} \qquad , \qquad k_{1_0} = \frac{k_0\, L_0^2}{T_0^2} \end{gathered} \tag{3.99}$$

Substituting (3.98) into (3.97) and then nondimensionalizing the resulting equations gives the following mathematical model.

$$\rho\, c_v \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} - k_1 \left(\frac{\partial T}{\partial x} \right)^2 \frac{\partial^2 T}{\partial x^2} = 0 \tag{3.100}$$

The left end of the bar is insulated and the right end of the bar is subjected to a temperature distribution that increases from T_1 to T_2 over $0 \leq t \leq \Delta t$ in continuous and differentiable manner and remains $T = T_2$ for $t \geq \Delta t$ (figure 3.1). Figure 3.1 also shows a schematic, boundary conditions and initial condition using dimensionless quantities.

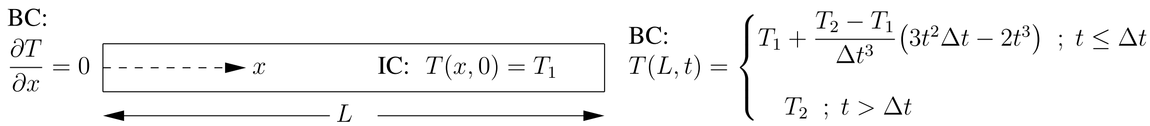


Figure 3.1: Schematic of 1-D heat transfer in a thermoelastic solid

Solution of the IVP:

A theoretical solution (3.100) is not readily possible. We consider numerical solutions of (3.100) using space-time least squares finite element processes based on residual functional with space-time local approximations in $H^{k,p}(\bar{\Omega}^e_{xt})$ spaces. The resulting non-linear algebraic equations from the space-time least squares process are solved using Newton's linear method. The computational processes in this approach are unconditionally stable throughout the evolution and permit higher order global differentiability local approximations. See reference [61–63] for details of local approximations and the least squares process for non-linear PDEs and higher order spaces. In the computations of the numerical solutions we choose

$$L_0 = \hat{L} = 0.1 \text{ m} \quad ; \quad T_0 = \hat{T}_1 = 300 \text{ K} \quad ; \quad \rho_0 = \hat{\rho} = 2700 \text{ kg/m}^3$$
$$c_{v_0} = \hat{c}_v = 903.0 \text{ J/kg.K} \quad ; \quad \hat{k}_0 = \hat{k} = 237 \text{ Watt/m.K}$$

which gives

$$L = 1 \quad ; \quad T_1 = 1 \quad ; \quad \rho = 1 \quad ; \quad c_v = 1 \quad ; \quad k = 1 \quad ; \quad k_1 = 39,974.68\,\hat{k}_1$$

A good discretization of the spatial domain $0 \leq x \leq 1$ is important in ensuring satisfactory convergence of the Newton's linear method for the system of non-linear algebraic equations and good accuracy of the computed solutions. A ten element uniform mesh with element length of 0.1 in space and time step of $\Delta t = 0.1$ is used in the present study. The local approximations are p-version (9-node elements) in higher order spaces. Initial p-convergence studies with this discretization suggest that $p = 12$ with $k = 2$, local approximations of class $C^{1,1}(\bar{\Omega}^e_{xt})$, to be sufficient for good accuracy of results. The residual or least squares functional values remain $O(10^{-7})$ - $O(10^{-14})$ indicating that the PDEs are satisfied very accurately. Newton's linear method used for solving the non-linear algebraic equations converges in less than 7 iterations for all results.

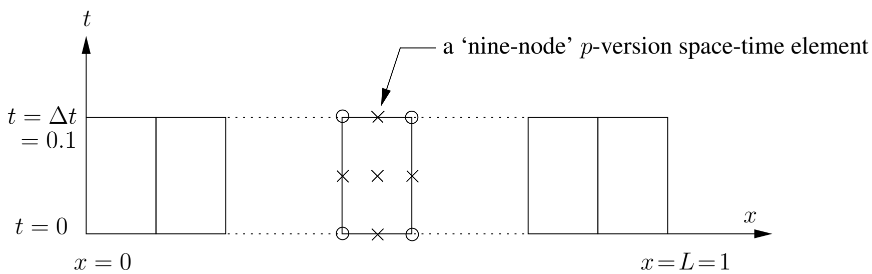


Figure 3.2: Uniform space-time discretization of the first space-time strip using ten 'nine node' p-version space-time elements

In the numerical studies we consider the following dimensionless temperature values

$$T_2 = 1.2 \quad ; \quad T_2 = 1.3 \quad ; \quad T_2 = 1.7 \quad ; \quad T_2 = 2.0 \tag{3.101}$$

which corresponds to $\hat{T}_2$ values of 360 K (86.85 °C), 390 K (116.85 °C), 510 K (236.85 °C) and 600 K (326.85 °C). We choose the following values of the coefficient $\hat{k}_1$.

$$\hat{k}_1 = 0.00\text{W.m/K}^3 \quad ; \quad \hat{k}_1 = 0.30\text{W.m/K}^3 \quad ; \quad \hat{k}_1 = 0.60\text{W.m/K}^3 \tag{3.102}$$

When $\hat{k}_1 = 0.00$, (3.100) reduces to the standard Fourier heat conduction equation. The corresponding values of k_1 are 0.00, 11392 and 22785.

The material coefficient $\hat{k}_1$ (or k_1) in the heat conduction law in (3.98) needs to be determined experimentally. The choice of the values of $\hat{k}_1$ is made so that with progressively increasing T_2, hence increasing temperature gradient, the Fourier heat conduction law remains valid up to some maximum value of T_2 i.e., for this range of T_2, the choices of the numerical values of k_1 do not influence heat conduction. Beyond a certain value of T_2, the numerical solutions obtained for non-zero k_1 begin to differ from those obtained using Fourier heat conduction law. We present numerical results in the following to demonstrate this.

Figures 3.3 - 3.6 show graphs of evolution of temperature T versus distance x for different values of T_2 and k_1. For T_2 of 1.2 and 1.3, the results corresponding to all three values of k_1 are in good agreement (figures 3.3 and 3.4) confirming that within this range of T_2, the Fourier heat conduction law holds as non-zero k_1 does not influence heat conduction.

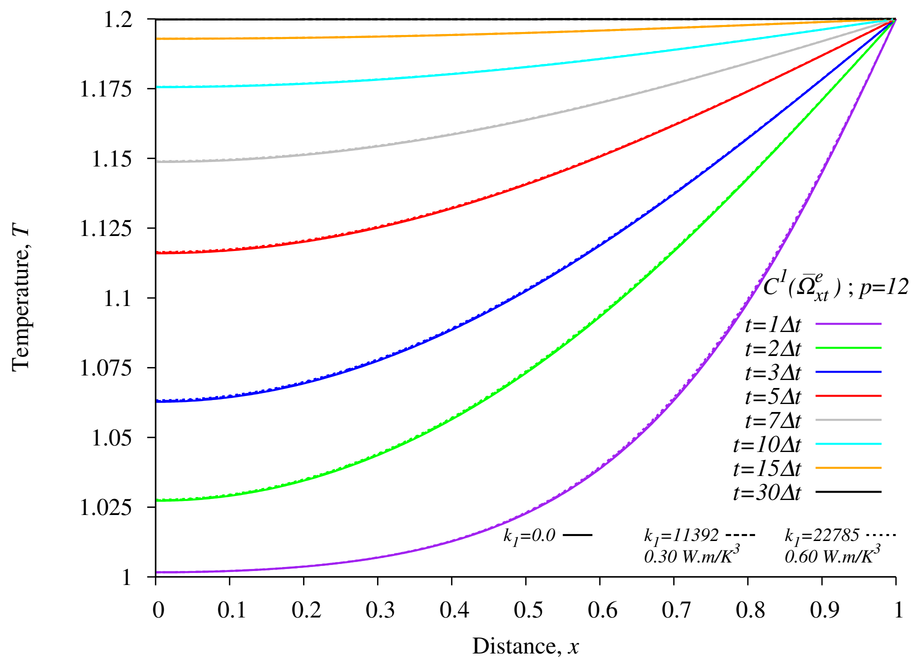


Figure 3.3: Temperature T versus distance x for a prescribed $T_2 = 1.2$

For $T_2 = 1.7$, the temperature distribution along the rod begins to differ from Fourier heat conduction law (figure 3.5). As expected, the larger the value of k_1, the greater is the deviation of the temperature distribution along the length of the rod from Fourier heat conduction law.

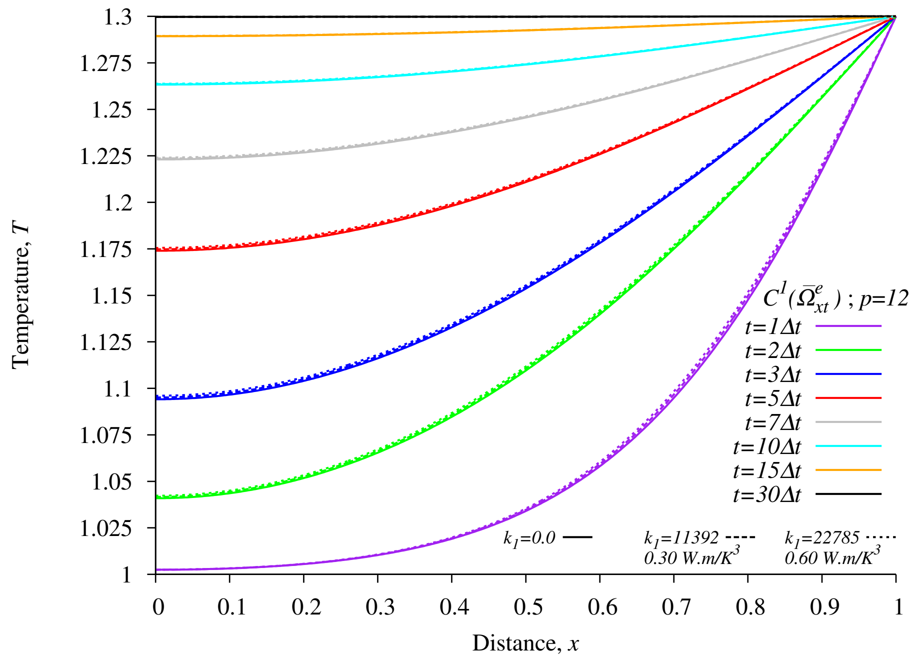


Figure 3.4: Temperature T versus distance x for a prescribed $T_2 = 1.3$

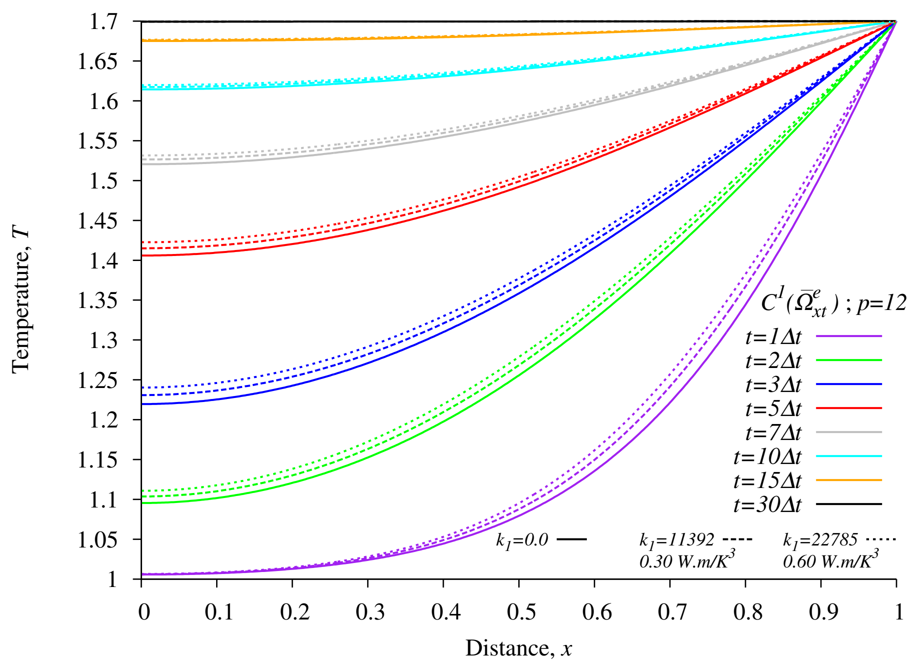


Figure 3.5: Temperature T versus distance x for a prescribed $T_2 = 1.7$

For $T_2 = 2.0$ (figure 3.6), the temperature distribution along the length of the rod differs significantly compared to Fourier heat conduction law. Larger value of k_1 results in greater deviation. The study demonstrates that when temperature gradients are high, the non-linear constitutive theory for the heat vector may be a more realistic representation of the physics as opposed to Fourier heat conduction law.

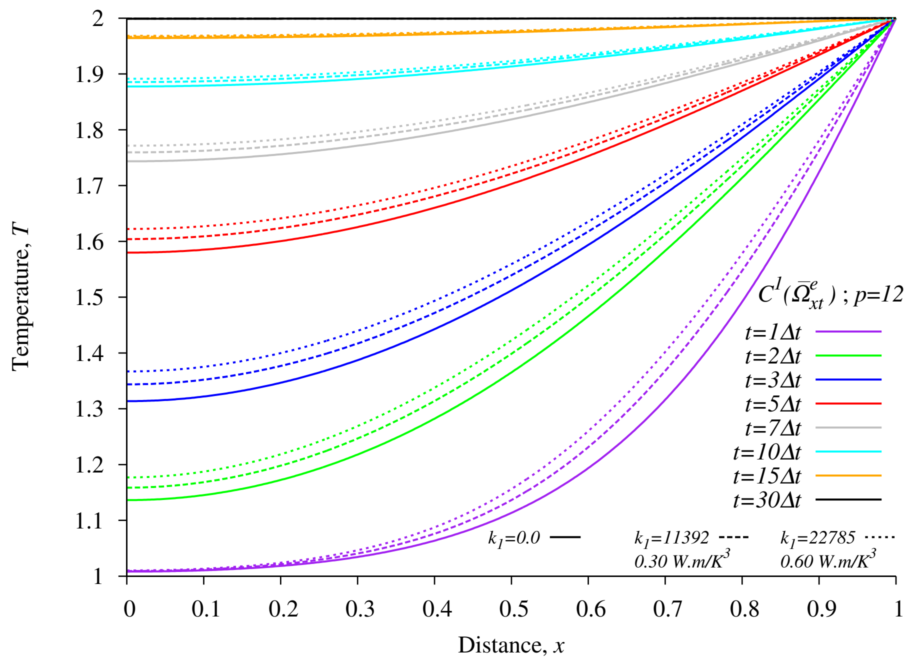


Figure 3.6: Temperature T versus distance x for a prescribed $T_2 = 2.0$

3.12 Summary

The rate constitutive theories in the Eulerian description for incompressible as well as compressible ordered thermoelastic solids have been presented in contravariant and covariant bases as well as using the Jaumann rates. When the mathematical models for deforming solids are constructed using the Eulerian description, the displacements of the material particles, and hence strain measures, are not readily obtainable. Thus the constitutive theories expressing chosen stress measures as a function of the conjugate strain measure is not usable. Hence, in this situation, one must consider a relationship between conjugate pairs of stress and strain rates, thus the need for rate constitutive theories.

Based on the axiom of admissibility, all constitutive equations must satisfy conservation laws to ensure thermodynamic equilibrium of the deforming matter. Since conservation of mass, balance of momenta and energy equation only require existence of the stress field and heat vector, these are independent of the constitution of the matter. Thus the second law of thermodynamics (the Clausius-Duhem inequality) must provide the basis for the constitutive theory. The conditions resulting from the Clausius-Duhem inequality show that η, specific entropy is deterministic from the Helmholtz free energy and hence should not be considered as a dependent variable in the constitutive theory, thus the Cauchy stress tensor, heat vector and the Helmholtz free energy density are the only dependent variables in the constitutive theory for the type of matter considered here. The conditions resulting from the entropy inequality also provide a mechanism to determine the heat vector as a function of the temperature gradient vector and conductivity, i.e., Fourier heat condition law. However, these conditions do not provide a mechanism to determine the constitutive theory for the total Cauchy stress tensor. If the total Cauchy stress tensor is decomposed into equilibrium stress and deviatoric stress, then the equilibrium stress is deterministic from the entropy inequality and leads to thermodynamic pressure for compressible matter and mechanical pressure in the case of incompressible matter when incompressibility constraint is incorporated in the entropy inequality. These hold regardless of the order of the rate constitutive theory. But the deviatoric Cauchy stress is not deterministic from the entropy inequality, however the entropy inequality does require the work expanded due to the deviatoric Cauchy stress to be positive. Thus the rate constitutive theory for ordered thermoelastic solids reduces to the deviatoric Cauchy stress tensor, heat vector and the Helmholtz free energy density as dependent variables and determination of the theory for them in contravariant, covariant bases and the Jaumann rates using the argument tensors describing the physics of deformation.

Details of the contra- and co-variant bases, stress and strain measures, convected time derivatives of the stress and strain tensors in contra- and co-variant bases, the Jaumann stress and strain rates, derivations of entropy inequality and the conditions resulting from it have been presented in references [55, 56]. It is shown that for compressible ordered thermoelastic solids, in contravariant basis, the argument tensors of the first convected time derivative of the deviatoric Cauchy stress $[{}_d\bar{\sigma}^{(0)}]$, i.e., $[{}_d\bar{\sigma}^{(1)}]$ and the heat vector $\bar{\boldsymbol{q}}^{(0)}$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[\gamma^{(j)}]$; $j = 1, 2, \ldots, n$, the convected time derivatives of orders $1, 2, \ldots, n$ in the contravariant basis and for $\bar{\Phi}$, the argument tensors are $\bar{\rho}$ and $\bar{\theta}$. In covariant basis, the argument tensors of the first convected time derivative of the deviatoric Cauchy stress $[{}_d\bar{\sigma}_{(0)}]$, i.e., $[{}_d\bar{\sigma}_{(1)}]$ and the heat vector $\bar{\boldsymbol{q}}_{(0)}$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[\gamma_{(j)}]$; $j = 1, 2, \ldots, n$, the convected time derivatives of orders $1, 2, \ldots, n$ in the covariant basis. The argument tensors for $\bar{\Phi}$ are $\bar{\rho}$ and $\bar{\theta}$. When using the Jaumann rates, we consider $[{}^{(0)}_{d}\bar{\sigma}^J]$ and ${}^{(0)}\bar{\boldsymbol{q}}^J$ as dependent variables in the constitutive theory with $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[{}^{(j)}\gamma^J]$; $j = 1, 2, \ldots, n$ as their argument tensors. For $\bar{\Phi}$, the argument tensors are $\bar{\rho}$ and $\bar{\theta}$. For incompressible ordered thermoelastic solids, density $\bar{\rho}$ in the current configuration is the same as in the reference configuration and hence it is no longer an argument of the dependent variables in the constitutive theory. Other arguments remain the same as for the compressible case.

The theory of generators and invariants is utilized to derive the general form of the constitutive equations for an n^{th} order 'ordered thermoelastic solid' (both compressible and incompressible) in contravariant and covariant bases as well as using the Jaumann rates. In this theory, both the first convected time derivative of the deviatoric Cauchy stress tensor and heat vector are expressed as a linear combination of the combined generators of the argument tensors. The coefficients in this linear combination are functions of the combined invariants of the argument tensors in addition to $\bar{\rho}$ and $\bar{\theta}$ (in case of compressible solids) or $\bar{\theta}$ (in case of incompressible solids) in the current configuration at time $t = t_{n+1}$. The coefficients in the linear combinations are determined by using their Taylor series expansion about a known configuration at time $t = t_n$ and retaining only up to linear terms in the combined invariants and the temperature. Explicit details are presented for second order 'ordered thermoelastic solids'. The general form of the constitutive equations are specialized and detailed derivations are presented for thermoelastic solids of order two and one as well as generalized hypo-thermoelastic solids and hypo-thermoelastic solids, both with variable material coefficients.

We note that the rate constitutive theories derived here for an ordered thermoelastic solid of order 'n' expresses the first convected time derivative of the deviatoric Cauchy stress tensor as a function of density $\bar{\rho}$, temperature $\bar{\theta}$, temperature gradient $\bar{\boldsymbol{g}}$ and the convected time derivatives of the conjugate strain tensor of up to order 'n' in a chosen basis, i.e., contra- or co-variant or the Jaumann. The contravariant basis yields upper convected rate constitutive equations whereas covariant basis gives lower convected rate constitutive equations. Likewise, use of the Jaumann rates yields the Jaumann rate constitutive equations. Surana et al. [30] have shown that in the case of finite deformation, only upper convected rate constitutive theory is in conformity with the physics of deformation. Based on the rate constitutive theories presented in this chapter for ordered thermoelastic solids we make the following specific remarks.

1. For ordered thermoelastic solids of order greater than or equal to two (i.e., when $[\gamma^{(2)}]$, $[\gamma^{(3)}]$, ... or $[\gamma_{(2)}]$, $[\gamma_{(3)}]$, ... or $[{}^{(2)}\gamma^J]$, $[{}^{(3)}\gamma^J]$, ... as argument tensors in addition to first convected time derivatives of the corresponding strain tensors), the contra- and co-variant stress measures as well as the Jaumann stress tensor are not the same even though they all are in x-frame with the same dyads. The same is true for the constitutive theory for the heat vector when the convected derivatives of the strain tensor of orders higher than one are the argument tensors.

2. Definition of $[{}_d\bar{\sigma}^{(1)}]$, $[{}_d\bar{\sigma}_{(1)}]$ and $[{}^{(1)}_{\ d}\bar{\sigma}^J]$ differ from each other. Furthermore, each of the three convected time derivatives also has a different definition for compressible and incompressible case. Thus, even in case of $[\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}]$ as the only argument tensor (in addition to $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$) of $[{}^{(1)}_{\ d}\bar{\sigma}]$, the resulting rate constitutive theories in contra- and covariant bases and using the Jaumann rates would be different. As shown [30], only contravariant

basis is physical when the deformation is finite.

3. When $[\bar{D}]$ is the only argument tensor of ${}^{(0)}\bar{\boldsymbol{q}}$ (in addition to $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$), then of course $\bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}_{(0)} = {}^{(0)}\bar{\boldsymbol{q}}^J$ i.e., the constitutive theory for the heat vector is independent of the basis.

4. The derivations of the constitutive theory for generalized compressible and incompressible hypo-thermoelastic solids (a subset of the rate constitutive theory of order one ($n = 1$)) allow us to define more complex and variable behavior of material coefficients during the evolution as they can be functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$, $(i_{{}^{(1)}\gamma})_{t_n}$, $(ii_{{}^{(1)}\gamma})_{t_n}$ and $(iii_{{}^{(1)}\gamma})_{t_n}$.

5. When using the theory of generators and invariants, the constitutive equation for the heat vector for an ordered thermoelastic solid is much more complex (even for thermoelastic solids of order one due to the dependence of the heat vector on the combined generators of $[\gamma^{(1)}]$, $\bar{\boldsymbol{g}}$ or $[\gamma_{(1)}]$, $\bar{\boldsymbol{g}}$ or $[{}^{(1)}\gamma^J]$, $\bar{\boldsymbol{g}}$) compared to Fourier heat conduction law which requires that the heat vector not be dependent on $[\gamma^{(1)}]$ or $[\gamma_{(1)}]$ or $[{}^{(1)}\gamma^J]$. The constitutive equation for the heat vector based on the combined generators of $[\gamma^{(1)}]$, $\bar{\boldsymbol{g}}$ or $[\gamma_{(1)}]$, $\bar{\boldsymbol{g}}$ or $[{}^{(1)}\gamma^J]$, $\bar{\boldsymbol{g}}$ is perhaps more realistic for finite deformation of solids as it accounts for velocity gradients.

6. When the first order rate constitutive equations ($n = 1$) are simplified to obtain constitutive equations for what is commonly known as hypo-elastic material, the restriction of infinitesimal deformation must be observed. To be more precise, in this case, second and higher order terms in the components of the first convected time derivatives of the strain tensor are assumed negligible. Thus, use of such constitutive relations [36, 37] for finite deformation may be questionable.

7. We point out that the Jaumann rate constitutive equations are probably most widely used for deforming solid matter in the Eulerian description [36, 37]. Surana et al. [30, 55, 56] have shown that the Jaumann rate constitutive equations are average of the constitutive equations in contra- and co-variant descriptions when the velocity field is the same in both bases. This is only true if the deformation is not finite. Nonetheless, these have been used widely for

finite deformation [36, 37].

8. A significant point to note in the present work is that determination of coefficients used in the linear combination of the generators to express the deviatoric stress tensor or heat vector requires use of Taylor series about the configuration at $t = t_n$ when $t = t_{n+1}$ is the current configuration. This automatically forces the determination of the coefficients in the configuration at time $t = t_n$ and not at $t = t_{n+1}$ corresponding to the current configuration. In the majority of the published works, this is not the case. Variable transport properties as well as coefficients dependent on ii_D are all expressed using the current configuration. This may be justified when the configurations at $t = t_n$ and $t = t_{n+1}$ are in close proximity in terms of deformation field but can not be supported by the derivations presented in this work.

9. The condition of positive work expanded due to deviatoric Cauchy stress tensor resulting from the entropy inequality must be satisfied by all rate constitutive equations. The work on constitutive inequalities supporting this condition is currently in progress.

10. The constitutive theories in this chapter is based on combined generators and invariants of the argument tensors of the dependent variables. These theories have continuum mechanics foundation as they satisfy the axioms of constitutive theory.

11. Applications of the simple rate constitutive theories resulting from the present work, such as constitutive theories for hypo-elastic solid have been presented in references [30, 64] for model problems consisting of fluid-solid interactions.

12. Numerical studies are presented to determine the influence of non-linearity in terms of temperature gradient in the constitutive equation for the heat vector and comparisons are made with the commonly used Fourier heat conduction law. Heat transfer in an incompressible thermoelastic solid (aluminum rod) under small motion and infinitesimal deformation was considered. In this case, the distinction between covariant and contravariant basis disappears. The new coefficient k_1 associated with the non-linear term must be determined experimen-

tally.

13. The rod has an uniform initial temperature T_1. The left end of the rod is insulated and the right end of the rod is subjected to a temperature distribution that increases from T_1 to T_2 over $0 \leq t \leq \Delta t$ in continuous and differentiable manner and remains $T = T_2$ for $t \geq \Delta t$. The choice of the values of k_1 were made so that with progressively increasing T_2, hence increasing temperature gradient, the Fourier heat conduction law remains valid up to some maximum value of T_2 i.e., for this range of T_2, the choices of the numerical values of k_1 do not influence heat conduction. Beyond a certain value of T_2, the numerical solutions obtained for non-zero k_1 begin to differ from those obtained using Fourier heat conduction law. The study demonstrated that when temperature gradients are high, the non-linear constitutive theory for the heat vector may be a more realistic representation of the physics as opposed to Fourier heat conduction law.

The work presented in this chapter provides completely general and unified theories for ordered thermoelastic solids from which specialized solid behaviors such as generalized hypo-thermoelastic solids and hypo-thermoelastic solids with variable material coefficients can be easily derived as shown in this chapter. It is demonstrated that the distinction between contra- and co-variant bases and Jaumann rates is critical for ordered thermoelastic solids regardless of the order of the solid.

Chapter 4

Rate Constitutive Theories in Eulerian Description for Ordered Thermofluids

4.1 Introduction

In this chapter we consider developments of rate constitutive theories for compressible as well as incompressible homogeneous and isotropic ordered thermofluids in Eulerian description in which the chosen deviatoric Cauchy stress tensor and the heat vector are functions of density, temperature, temperature gradient and the convected time derivatives of the conjugate strain tensors of up to a desired order 'n'. The fluids described by these constitutive theories will be referred to as *ordered thermofluids* due to the fact that the constitutive theories for the deviatoric Cauchy stress tensor and heat vector are dependent on the order 'n' of the convected time derivatives of the conjugate strain tensor. The highest order of the convected time derivative of the strain tensor defines the order of the thermofluid.

We have intentionally used the term 'thermofluids' as opposed to 'thermoviscous fluids' due to the fact that the constitutive theories presented here describe a broader group of fluids than Newtonian and generalized Newtonian fluids that are commonly referred as thermoviscous fluids.

Newton's law of viscosity for incompressible and compressible fluids are well known and widely used as constitutive equations for incompressible and compressible thermoviscous fluids (Newtonian fluids) [38, 39]. The constitutive models for generalized Newtonian fluids, such as power law and Carreau-Yasuda model, are extensions of the constitutive models for Newtonian fluids in which the medium viscosity is assumed to depend on the deformation field [22].

The developments of general ordered rate constitutive theories are presented in contravariant and covariant bases as well as using Jaumann rates based on the principles and axioms of continuum mechanics. The general ordered rate constitutive theories are also simplified to obtain the constitutive equations for the well known generalized Newtonian and Newtonian fluids. The developments of rate constitutive theories using contravariant, covariant and Jaumann bases and the consequences of the choice of basis are discussed and illustrated in the general derivation as well as specialized cases.

4.2 Rate Constitutive Theories in Eulerian Description

In chapter 2, choices of the dependent variables for the constitutive theories in Eulerian description based on entropy inequality were considered. For these choices of dependent variables, three possibilities were discussed for their argument tensors. In chapter 3, a summary of these has been presented for rate constitutive theories for thermoelastic solids. In this section also, we present a similar summary with appropriate modifications followed by the specific choices of the dependent variables and their argument tensors for rate constitutive theories for thermofluids in Eulerian description. In chapter 2 we had considered entropy inequality in Lagrangian description to conclude that Φ, $\boldsymbol{\sigma}^*$, $\boldsymbol{q}$ and η must be the dependent variables in the constitutive theories. We considered $[J]$, $[\dot{J}]$, θ and $\boldsymbol{g}$ as arguments of the dependent variables in the constitutive theories. Using entropy inequality in Lagrangian description it was concluded that: (i) Φ is not a function of $[\dot{J}]$ (ii) Φ is not a function of $\boldsymbol{g}$ either (iii) η is not a dependent variable in the constitutive theory

(iv) consideration of $\rho_0 \frac{\partial\Phi}{\partial J_{ik}} - \sigma^*_{ki} = 0$ and $\underset{\sim}{q_i}\underset{\sim}{g_i} \leq 0$ is inappropriate due to the fact that in this case $\boldsymbol{\sigma}^*$ is not a function of $[\dot{J}]$ as Φ is not a function of $[\dot{J}]$, which is contrary to the assumption that $\boldsymbol{\sigma}^*$ depends on $[\dot{J}]$. Thus, entropy inequality does not provide any further means of determining the constitutive theories for neither $\boldsymbol{\sigma}^*$ nor $\boldsymbol{q}$. It was shown that by considering stress decomposition into equilibrium and deviatoric stress i.e. $\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^* + {}_d\boldsymbol{\sigma}^*$ in which ${}_e\boldsymbol{\sigma}^*$ is not a function of $[\dot{J}]$ and ${}_d\boldsymbol{\sigma}^*$ becomes zero when $[\dot{J}]$ and $\boldsymbol{g}$ are zero, and using the conditions resulting from the entropy inequality, that ${}_d\sigma^*_{ki}\dot{J}_{ik} > 0$ and $\underset{\sim}{q_i}\underset{\sim}{g_i} \leq 0$ must hold, which gave us $\Phi = \Phi([J], \theta)$, $\sigma^*_{ij} = \rho_0 \frac{\partial\Phi}{\partial J_{ji}} + {}_d\sigma^*_{ij}([J], [\dot{J}], \theta, \boldsymbol{g})$ and $\boldsymbol{q} = \boldsymbol{q}([J], [\dot{J}], \theta, \boldsymbol{g})$. Due to frame invariance considerations, dependence on $[J]$ must be replaced by I_J, $I\!I_J$, $I\!I\!I_J$ and $[\dot{J}]$ can be replaced by I_J, $I\!I_J$, $I\!I\!I_J$, $[D]$. These arguments hold in Lagrangian description. However, in Eulerian description, material point displacements are not known, hence $[J]$ is not deterministic but $I\!I\!I_J = \det[J] = \rho_0/\bar{\rho}$ i.e. dependence on $I\!I\!I_J$ can be replaced by $\bar{\rho}$, but dependence on I_J and $I\!I_J$ can not be considered. Thus, in Eulerian description we consider the following in contravariant basis

$$
\begin{aligned}
&\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \\
&[\bar{\sigma}^{(0)}] = [\,{}_e\bar{\sigma}^{(0)}\,] + [\,{}_d\bar{\sigma}^{(0)}(\bar{\rho}, [\bar{D}], \bar{\theta}, \bar{\boldsymbol{g}})\,] \quad ; \quad [{}_e\sigma^*]^T = \rho_0 \frac{\partial\Phi}{\partial[J]} \\
&\bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}^{(0)}(\bar{\rho}, [\bar{D}], \bar{\theta}, \bar{\boldsymbol{g}})
\end{aligned}
\tag{4.1}
$$

For compressible matter, equilibrium stress is a function $\bar{\Phi}$ and thus it is deterministic from the deformation field. For incompressible matter, equilibrium stress is also derived from the entropy inequality in conjunction with incompressibility constraint, however, equilibrium stress is not a function of $\bar{\Phi}$ and thus it is not deterministic from the deformation field. It was shown that in both cases, equilibrium stress is independent of the basis. We make the following remarks:

(1) The second law of thermodynamics only restricts the work expanded due to the deviatoric stress to be positive but provides no mechanism for determining the constitutive theory for the deviatoric stress. In addition, $\underset{\sim}{q_i}\underset{\sim}{g_i} \leq 0$ must also hold.

(2) The theory of generators and invariants [3–21] provides a continuum mechanics foundation

to derive constitutive equations for the deviatoric Cauchy stress tensor and heat vector in which we determine combined generators of the argument tensors that form *integrity or minimal basis*. The dependent variables in the constitutive theories are expressed as linear combinations of the combined generators of the argument tensors. The coefficients used in the linear combinations are functions of $\bar{\rho}$, $\bar{\theta}$, and the combined invariants of the argument tensors in the current configuration which, using the *axiom of smooth neighborhood*, are determined by using their Taylor series expansion about a previously known configuration.

4.3 Thermofluids: dependent variables in the constitutive theories and their argument tensors

Let $[\gamma^{(j)}]$, $[\gamma_{(j)}]$, $[{}^{(j)}\gamma^J]$; $j = 1, 2, \ldots, n$ be the convected time derivatives of order $1, 2, \ldots, n$ of the Almansi strain tensor $[\bar{\varepsilon}]$, Green's strain tensor $[\varepsilon]$ and Jaumann strain rates. These are *fundamental kinematic symmetric tensors of rank two*. Likewise, let $[{}_d\bar{\sigma}^{(k)}]$, $[{}_d\bar{\sigma}_{(k)}]$ and $[{}^{(k)}_{\,d}\bar{\sigma}^J]$; $k = 0, 1, \ldots, m$ be convected time derivatives of orders $0, 1, \ldots, m$ of the Cauchy stress tensors in the three bases. These are *fundamental symmetric tensors of rank two*. We note that $[\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}]$. Using the new notation introduced in chapter 2 we can generalize (4.1) by replacing $[\bar{D}]$ with $[{}^{(j)}\gamma]$; $j = 1, 2, \ldots, n$ depending upon whether the basis is contra- or co-variant or Jaumann basis.

Consider current configuration at time $t = t_{n+1}$. In the rate constitutive theories presented in this chapter for thermofluids, we consider the deviatoric Cauchy stress tensor as dependent variable in the constitutive theories. Thus, $\bar{\Phi}(\cdot)$, Helmholtz free energy density; $[{}^{(0)}_{\,d}\bar{\sigma}\,]$, deviatoric Cauchy stress tensor, and ${}^{(0)}\bar{\boldsymbol{q}}$, the heat vector, are dependent variables in the constitutive theories. $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$, $[{}^{(j)}\gamma\,]$; $j = 1, 2, \ldots, n$ appear (as appropriate) as their argument tensors. Hence, we have the following for compressible and incompressible thermofluids (see chapter 2, choice I).

Compressible thermofluids:

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\big) \tag{4.2}$$

$$[{}^{(0)}\bar{\sigma}\ ] = \big[{}^{(0)}_{e}\bar{\sigma}\big(\bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\big)\big] + [{}^{(0)}_{d}\bar{\sigma}\ ] \tag{4.3}$$

$$[{}^{(0)}_{d}\bar{\sigma}\ ] = \big[{}^{(0)}_{d}\bar{\sigma}\big(\bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j=1,2,\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big)\big] \tag{4.4}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ \bar{\rho}(\bar{\boldsymbol{x}},t)\ ,\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j=1,2\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big) \tag{4.5}$$

in which equilibrium stress $[{}^{(0)}_{e}\bar{\sigma}]$ is thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})[I]$ and it is independent of the basis. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\rho},\bar{\theta})$ can be replaced by $-\bar{p}(\bar{\rho},\bar{\theta})$ (see chapter 2, section 2.6 for derivation).

Incompressible thermofluids:

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\theta}(\bar{\boldsymbol{x}},t)\big) \tag{4.6}$$

$$[{}^{(0)}\bar{\sigma}\ ] = \big[{}^{(0)}_{e}\bar{\sigma}\big(\bar{\theta}(\bar{\boldsymbol{x}},t)\big)\big] + [{}^{(0)}_{d}\bar{\sigma}\ ] \tag{4.7}$$

$$[{}^{(0)}_{d}\bar{\sigma}\ ] = \big[{}^{(0)}_{d}\bar{\sigma}\big(\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j=1,2,\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big)\big] \tag{4.8}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\ ;\ j=1,2\dots,n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}},t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big) \tag{4.9}$$

where equilibrium stress $[{}^{(0)}_{e}\bar{\sigma}]$ is mechanical pressure $\bar{p}(\bar{\theta})[I]$ and it also is independent of the basis. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\theta})$ can be replaced by $-\bar{p}(\bar{\theta})$ (see chapter 2, section 2.6 for derivation).

The rate constitutive theories of various orders for $[{}^{(0)}_{d}\bar{\sigma}\ ]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ derived using (4.4) and (4.5), or (4.8) and (4.9) for compressible and incompressible case can be converted to contravariant basis, covariant basis or the Jaumann rates by choosing $[{}^{(0)}_{d}\bar{\sigma}\ ]$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma\ ]$; $j=1,2,\dots,n$ as $([{}_{d}\bar{\sigma}^{(0)}]$,

$\bar{\boldsymbol{q}}^{(0)}$ and $[\gamma^{(j)}]$; $j = 1, 2, \dots, n$) or ($[{}_d\bar{\sigma}_{(0)}]$, $\bar{\boldsymbol{q}}_{(0)}$ and $[\gamma_{(j)}]$; $j = 1, 2, \dots, n$) or ($[{}^{(0)}_{d}\bar{\sigma}^J]$, ${}^{(0)}\bar{\boldsymbol{q}}^J$ and $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$).

In the following sections we consider details of the derivations of rate constitutive theories in Eulerian description for the deviatoric Cauchy stress tensor and heat vector for both compressible as well as incompressible thermofluids.

4.4 Rate constitutive theory of order '*n*': compressible thermofluids

Consider a deforming volume of compressible thermofluid at time $t = t_{n+1}$, the current configuration. We derive the rate constitutive theory of order 'n' for deviatoric Cauchy stress $[{}^{(0)}_{d}\bar{\sigma}]$ and heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ using ((4.4) and (4.5))

$$\begin{aligned} [{}^{(0)}_{d}\bar{\sigma}] &= [{}^{(0)}_{d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(j)}\gamma]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \\ {}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [{}^{(j)}\gamma]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \end{aligned} \tag{4.10}$$

4.4.1 Constitutive theory for the deviatoric Cauchy stress tensor

Let $[{}^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \dots, N$ be the combined generators of the argument tensors $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ and $\bar{\boldsymbol{g}}$ of $[{}^{(0)}_{d}\bar{\sigma}]$ that are symmetric tensors of rank two, and let ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$ be the combined invariants of the same argument tensors. Then, we can express $[{}_d\bar{\sigma}^{(0)}]$ as a linear combination of the generators $[{}^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \dots, N$ and the identity tensor $[I]$.

$$[{}^{(0)}_{d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + \sum_{i=1}^{N} {}^{\sigma}\alpha^i[{}^{\sigma}\underset{\sim}{G}^i] \tag{4.11}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, N$ in (4.11) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$,

$\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \ldots, M$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, N$ in (4.11) related to the configuration at time $t = t_{n+1}$ we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, N$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{M} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 0, 1, \ldots, N \quad (4.12) $$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \ldots, M$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \ldots, N$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$ whereas in (4.12), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, M, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \ldots, M\big)$; $i = 0, 1, \ldots, N$. When (4.12) is substituted in (4.11), we obtain the final expression for the most general rate constitutive theory of order n for $[{}^{(0)}_{\;\;d}\bar{\sigma}]$ for compressible thermofluids.

4.4.2 Constitutive theory for the heat vector

Let $\{{}^{q}\underset{\sim}{G}^i\}$; $i = 1, 2, \ldots, \widetilde{N}$ be the combined generators of the argument tensors $[{}^{(j)}\gamma]$; $j = 1, 2, \ldots, n$ and $\bar{\boldsymbol{g}}$ of ${}^{(0)}\bar{\boldsymbol{q}}$ that are tensors of rank one. The combined invariants of the argument tensors obviously remain same as for $[{}^{(0)}_{\;\;d}\bar{\sigma}]$ i.e., ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M$. Then we can express ${}^{(0)}\bar{\boldsymbol{q}}$ as a linear combination of $\{{}^{q}\underset{\sim}{G}^i\}$; $i = 1, 2, \ldots, \widetilde{N}$.

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{\widetilde{N}} {}^{q}\alpha^i \{{}^{q}\underset{\sim}{G}^i\} \quad (4.13) $$

The absence of unit vector in (4.13) *as a generator is due to the fact that uniform temperature field does not contribute to* ${}^{(0)}\bar{\boldsymbol{q}}$. *The negative sign in* (4.13) *is because a positive* ${}^{(0)}\bar{\boldsymbol{q}}$ *in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter*. The coefficients ${}^{\sigma}\alpha^i$; $i = 1, 2, \ldots, \widetilde{N}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j; j = 1, 2, \ldots, M$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \ldots, M$. To determine the coefficients ${}^{q}\alpha^i$; $i = 1, 2, \ldots, \widetilde{N}$ (in the current configuration at time $t = t_{n+1}$) in (4.13), we consider Taylor series expansion of each ${}^q\!\alpha^i$; $i = 1, 2, \dots, \widetilde{N}$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^q\!\alpha^i = {}^q\!\alpha^i\Big|_{t_n} + \sum_{j=1}^{M} \frac{\partial({}^q\!\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\Bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^q\!\alpha^i)}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 1, 2, \dots, \widetilde{N} \tag{4.14}
$$

${}^q\!\alpha^i\big|_{t_n}$, $\frac{\partial({}^q\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, M$ and $\frac{\partial({}^q\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1, 2, \dots, \widetilde{N}$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, M$ whereas in (4.14), ${}^q\!\alpha^i = {}^q\!\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, M, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \dots, M\big)$; $i = 1, 2 \dots, \widetilde{N}$. When (4.14) is substituted in (4.13), we obtain the final expression for the most general rate constitutive theory of order n for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermofluids.

4.4.3 Remarks:

1. In sections 4.4.1 - 4.4.2 we have presented n^{th} order rate constitutive theories for the deviatoric Cauchy stress tensor and the heat vector using $[{}^{(0)}_{\ d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ as dependent variables with $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ strain rate tensors as their argument tensors, in addition to $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$. Hence, these developments are independent of the basis.

2. By replacing $[{}^{(0)}_{\ d}\bar{\sigma}]$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ with the appropriate corresponding measures in the chosen basis, we can readily obtain the n^{th} order rate theories for the deviatoric Cauchy stress tensor and heat vector in the desired basis. More specifically we use the following measures:

$$
\begin{array}{lllllll}
\text{Contravariant basis:} & [{}_d\bar{\sigma}^{(0)}] & , & \bar{\boldsymbol{q}}^{(0)} & , & [\gamma^{(j)}] & ; \; j = 1, 2, \dots, n \\
\text{Covariant basis:} & [{}_d\bar{\sigma}_{(0)}] & , & \bar{\boldsymbol{q}}_{(0)} & , & [\gamma_{(j)}] & ; \; j = 1, 2, \dots, n \\
\text{Jaumann:} & [{}^{(0)}_{\ d}\bar{\sigma}^J] & , & {}^{(0)}\bar{\boldsymbol{q}}^J & , & [{}^{(j)}\gamma^J] & ; \; j = 1, 2, \dots, n
\end{array} \tag{4.15}
$$

3. Since the tensor $\bar{\boldsymbol{g}}$ is independent of the basis, the combined generators and the combined

invariants used in sections 4.4.1 - 4.4.2 only need to be redefined using the convected rates $[\gamma^{(j)}]$; $j = 1, 2, \dots, n$, $[\gamma_{(j)}]$; $j = 1, 2, \dots, n$ and $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$ for the contravariant, covariant and Jaumann n^{th} order rate theories.

4. In the final expression for $[{}^{(0)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ containing sum of many group of terms, we consider the following arrangement (in general).

   (a) In each group, *the terms that are defined in the configuration at time $t = t_n$ are grouped to define material coefficients*.

   (b) With choice (a), the expression for $[{}^{(0)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ will now consist of the sum of the material coefficients defined in the configuration at $t = t_n$ multiplied with the generators and/or invariants in the current configuration at time $t = t_{n+1}$ in which the deformation is not known.

   (c) The material coefficients defined in (a) will be functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, M$.

   We follow this arrangement (as far as possible) in all subsequent derivations. These theories use integrity and hence are complete but are too complicated and unpractical as they contain too many material coefficients that must be determined experimentally and/or empirically. Details of (a) - (c) are clearly shown in sections 4.7.1 and 4.7.2.

5. Dependence of the coefficients in the final form of the constitutive equations for deviatoric Cauchy stress tensor and heat vector on $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, M$ permits variable material coefficients during the evolution. Thus material coefficients can be a function of density and temperature during the evolution for which experimental and/or empirical relations such as power law, Sutherland law etc. are justified. Furthermore, dependence of the coefficients on the invariants $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, M$ permits complex description of material coefficients on the deformation field. Shear thinning, shear thickening behaviors described by power law, Carreau-Yasuda models etc. based on experiments and/or empirical relations are permissible within the framework of the theory presented here.

6. An important point to note is that *the material coefficients in the final form of the constitutive equations are defined using the configuration at time $t = t_n$ whereas the constitutive equations hold for the current configuration at time $t = t_{n+1}$*. This of course is a consequence of the Taylor series expansion of the coefficients in the linear combination using generators about the configuration at time $t = t_n$. In the currently used models in the published works [22, 38, 39] for variable material coefficients, the coefficients are expressed as a function of the unknown deformation field in the current configuration at time $t = t_{n+1}$. This is obviously not supported by the derivations of the constitutive theories presented here in sections 4.4.1 and 4.4.2.

4.5 Rate constitutive theory of order two (*n*=2): compressible thermofluids

If we limit the convected time derivatives of the strain tensor to just first and second as argument tensors in the constitutive theory, i.e., if we only consider $[^{(1)}\gamma]$ and $[^{(2)}\gamma]$ as argument tensors, then we can explicitly present specific forms and expressions for the combined generators and invariants in the constitutive theory. This theory defines thermofluids of order two.

$$\begin{aligned} [{}^{(0)}_{\ d}\bar{\sigma}] &= [{}^{(0)}_{\ d}\bar{\sigma}(\bar{\rho}\ ,\ [^{(1)}\gamma]\ ,\ [^{(2)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \\ {}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [^{(1)}\gamma]\ ,\ [^{(2)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \end{aligned} \tag{4.16}$$

4.5.1 Constitutive theory for the deviatoric Cauchy stress tensor

The combined generators $[^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \ldots, 12$ of the argument tensors $[^{(1)}\gamma]$, $[^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two are listed in table 4.1. The combined invariants ${}^{q\sigma}\underset{\sim}{I}^j; j = 1, 2, \ldots, 16$ of the tensors $[^{(1)}\gamma]$, $[^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ are listed in table 4.2 [3–21].

***Remarks*:**

(i) We note that the invariants listed in table 4.2 under (2) (marked (a)) need not be included due to the fact that

$$\text{tr}([^{(1)}\gamma][^{(2)}\gamma] + [^{(2)}\gamma][^{(1)}\gamma]) + \text{tr}([^{(1)}\gamma][^{(2)}\gamma] - [^{(2)}\gamma][^{(1)}\gamma]) = 2\text{tr}([^{(1)}\gamma][^{(2)}\gamma])$$

which is same as ${}^{q\sigma}\underset{\sim}{I}^{8}$ (except the factor 2, which is of no consequence). In many published works (a) are also included in addition to ${}^{q\sigma}\underset{\sim}{I}^{8}$ which is redundant [55].

(ii) Likewise, in many published works the invariant ${}^{q\sigma}\underset{\sim}{I}^{16}$ is replaced with the two invariants listed under item (3) (marked (b)). Following (i), the sum of the invariants marked (b) is two times ${}^{q\sigma}\underset{\sim}{I}^{16}$. Hence including these in place of ${}^{q\sigma}\underset{\sim}{I}^{16}$ is inappropriate as well.

Now we can express $[{}^{(0)}_{\;d}\bar{\sigma}]$ as a linear combination of $[I]$ and the generators $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \ldots, 12$

$$[{}^{(0)}_{\;d}\bar{\sigma}] = {}^{\sigma}\alpha^{0}[I] + \sum_{i=1}^{12} {}^{\sigma}\alpha^{i}[{}^{\sigma}\underset{\sim}{G}^{i}] \tag{4.17}$$

The coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \ldots, 12$ in (4.17) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \ldots, 16$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}$; $j = 1, 2, \ldots, 16$. To determine the coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \ldots, 12$ in (4.17) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \ldots, 12$ about the configuration at time $t = t_{n}$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \ldots, 16$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\Big|_{t_n} + \sum_{j=1}^{16} \frac{\partial({}^{\sigma}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\bigg|_{t_n} \Big(({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n} \Big) + \frac{\partial({}^{\sigma}\alpha^{i})}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 0, 1, \ldots, 12 \tag{4.18}$$

${}^{\sigma}\alpha^{i}\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\big|_{t_n}$; $j = 1, 2, \ldots, 16$ and $\frac{\partial({}^{\sigma}\alpha^{i})}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \ldots, 12$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}$; $j = 1, 2, \ldots, 16$ whereas in (4.18), ${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\big(\bar{\rho}_{t_n}\,,\, \bar{\theta}_{t_n}\,,\, ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}; j = 1, 2, \ldots, 16\,,\, \bar{\theta}_{t_{n+1}}\,,$ $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}; j = 1, 2, \ldots, 16\big)$; $i = 0, 1, \ldots, 12$. When (4.18) is substituted in (4.17), we obtain the

Table 4.1: Combined generators for $[{}^{(0)}_{\;d}\bar{\sigma}]$

	Arguments	**Generators**
(1)	none	$[I]$
(2)	one at a time (including (1))	
	$[{}^{(1)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^{1}] = [{}^{(1)}\gamma] \quad ; \quad [{}^{\sigma}\underset{\sim}{G}^{2}] = [{}^{(1)}\gamma]^2$
	$[{}^{(2)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^{3}] = [{}^{(2)}\gamma] \quad ; \quad [{}^{\sigma}\underset{\sim}{G}^{4}] = [{}^{(2)}\gamma]^2$
	$\bar{\boldsymbol{g}}$	$[{}^{\sigma}\underset{\sim}{G}^{5}] = \bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$
(3)	two at a time (including (1) and (2))	
	$[{}^{(1)}\gamma] \quad , \quad [{}^{(2)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^{6}] = [{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]$ $[{}^{\sigma}\underset{\sim}{G}^{7}] = [{}^{(1)}\gamma]^2[{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]^2$ $[{}^{\sigma}\underset{\sim}{G}^{8}] = [{}^{(1)}\gamma][{}^{(2)}\gamma]^2 + [{}^{(2)}\gamma]^2[{}^{(1)}\gamma]$
	$[{}^{(1)}\gamma] \quad , \quad \bar{\boldsymbol{g}}$	$[{}^{\sigma}\underset{\sim}{G}^{9}] = \bar{\boldsymbol{g}} \otimes [{}^{(1)}\gamma]\bar{\boldsymbol{g}} + [{}^{(1)}\gamma]\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$ $[{}^{\sigma}\underset{\sim}{G}^{10}] = \bar{\boldsymbol{g}} \otimes [{}^{(1)}\gamma]^2\bar{\boldsymbol{g}} + [{}^{(1)}\gamma]^2\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$
	$[{}^{(2)}\gamma] \quad , \quad \bar{\boldsymbol{g}}$	$[{}^{\sigma}\underset{\sim}{G}^{11}] = \bar{\boldsymbol{g}} \otimes [{}^{(2)}\gamma]\bar{\boldsymbol{g}} + [{}^{(2)}\gamma]\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$ $[{}^{\sigma}\underset{\sim}{G}^{12}] = \bar{\boldsymbol{g}} \otimes [{}^{(2)}\gamma]^2\bar{\boldsymbol{g}} + [{}^{(2)}\gamma]^2\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}$

most general second order ($n = 2$) rate constitutive theory for $[{}^{(0)}_{\;d}\bar{\sigma}]$ for compressible thermofluids. This theory uses integrity and hence is complete. We follow remarks in section 4.4.3 to define material coefficients in the final expression for $[{}^{(0)}_{\;d}\bar{\sigma}]$ and to obtain the corresponding rate theories in contravariant basis, covariant basis and using Jaumann rates.

Table 4.2: Combined invariants for $[{}^{(0)}_{\,d}\bar{\sigma}]$: Also valid for ${}^{(0)}\bar{\boldsymbol{q}}$

	Arguments	**Invariants**
(1)	one at a time	
	$[{}^{(1)}\gamma]$	${}^{q\sigma}\underset{\sim}{I}^{1} = \text{tr}([{}^{(1)}\gamma])$; ${}^{q\sigma}\underset{\sim}{I}^{2} = \text{tr}([{}^{(1)}\gamma]^2)$ ${}^{q\sigma}\underset{\sim}{I}^{3} = \text{tr}([{}^{(1)}\gamma]^3)$
	$[{}^{(2)}\gamma]$	${}^{q\sigma}\underset{\sim}{I}^{4} = \text{tr}([{}^{(2)}\gamma])$; ${}^{q\sigma}\underset{\sim}{I}^{5} = \text{tr}([{}^{(2)}\gamma]^2)$ ${}^{q\sigma}\underset{\sim}{I}^{6} = \text{tr}([{}^{(2)}\gamma]^3)$
	$\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{7} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$
(2)	two at a time $\big($ including (1) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$	${}^{q\sigma}\underset{\sim}{I}^{8} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma])$; ${}^{q\sigma}\underset{\sim}{I}^{9} = \text{tr}([{}^{(1)}\gamma]^2[{}^{(2)}\gamma])$ ${}^{q\sigma}\underset{\sim}{I}^{10} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma]^2)$; ${}^{q\sigma}\underset{\sim}{I}^{11} = \text{tr}([{}^{(1)}\gamma]^2[{}^{(2)}\gamma]^2)$ (a) $\boxed{\begin{array}{l} {}^{q\sigma}\underset{\sim}{I} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]) \\ {}^{q\sigma}\underset{\sim}{I} = \text{tr}([{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]) \end{array}}$
	$[{}^{(1)}\gamma]$, $\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{12} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}}$; ${}^{q\sigma}\underset{\sim}{I}^{13} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}}$
	$[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{14} = \bar{\boldsymbol{g}} \cdot [{}^{(2)}\gamma]\,\bar{\boldsymbol{g}}$; ${}^{q\sigma}\underset{\sim}{I}^{15} = \bar{\boldsymbol{g}} \cdot [{}^{(2)}\gamma]^2\,\bar{\boldsymbol{g}}$
(3)	three at a time $\big($ including (1) and (2) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	${}^{q\sigma}\underset{\sim}{I}^{16} = \bar{\boldsymbol{g}} \cdot [{}^{(1)}\gamma][{}^{(2)}\gamma]\,\bar{\boldsymbol{g}}$ (b) $\boxed{\begin{array}{l} {}^{q\sigma}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}} \\ {}^{q\sigma}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}} \end{array}}$

4.5.2 Constitutive theory for the heat vector

The combined generators $\{{}^{q}\underset{\sim}{G}^{i}\}$; $i = 1, 2, \dots, 7$ of the argument tensors $[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one are given in table 4.3 [3–21]. The combined invariants of the argument tensors obviously remain the same as listed in table 3.2 i.e., ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, 16$. Using the combined generators $\{{}^q\underset{\sim}{G}^i\}$; $i = 1, 2, \ldots, 7$ we can write

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{7} {}^q\alpha^i \{{}^q\underset{\sim}{G}^i\} \tag{4.19} $$

Table 4.3: Combined generators for ${}^{(0)}\bar{\boldsymbol{q}}$

	Arguments	**Generators**
(1)	one at a time	
	$[{}^{(1)}\gamma]$	none
	$[{}^{(2)}\gamma]$	none
	$\bar{\boldsymbol{g}}$	$\{{}^q\underset{\sim}{G}^1\} = \bar{\boldsymbol{g}}$
(2)	two at a time $\big($ including (1) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$	none
	$[{}^{(1)}\gamma]$, $\bar{\boldsymbol{g}}$	$\{{}^q\underset{\sim}{G}^2\} = [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}}$ $\{{}^q\underset{\sim}{G}^3\} = [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}}$
	$[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	$\{{}^q\underset{\sim}{G}^4\} = [{}^{(2)}\gamma]\,\bar{\boldsymbol{g}}$ $\{{}^q\underset{\sim}{G}^5\} = [{}^{(2)}\gamma]^2\,\bar{\boldsymbol{g}}$
(3)	three at a time $\big($ including (1) and (2) $\big)$	
	$[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$, $\bar{\boldsymbol{g}}$	$\{{}^q\underset{\sim}{G}^6\} = \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] + [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}}$ $\{{}^q\underset{\sim}{G}^7\} = \big[[{}^{(1)}\gamma][{}^{(2)}\gamma] - [{}^{(2)}\gamma][{}^{(1)}\gamma]\big]\,\bar{\boldsymbol{g}}$

The rational for omitting the unit vector and using negative sign in (4.19) has already been explained in section 4.4.2. The coefficients ${}^{\sigma}\alpha^i$; $i = 1, 2, \dots, 7$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j; j = 1, 2, \dots, 16$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \dots, 16$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 1, 2, \dots, 7$ (in the current configuration at time $t = t_{n+1}$) in (4.19), we consider Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 1, 2, \dots, 7$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 16$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{t_n} + \sum_{j=1}^{16} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \left(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\right) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 1, 2, \dots, 7 \quad (4.20) $$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, 16$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1, 2, \dots, 7$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, 16$ whereas in (4.20), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, 16, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \dots, 16\big)$; $i = 1, 2, \dots, 7$. When (4.20) is substituted in (4.19), we obtain the most general second order ($n = 2$) rate constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermofluids. In this case also, we follow remarks in section 4.4.3 to define material coefficients in the final expression for ${}^{(0)}\bar{\boldsymbol{q}}$ and to obtain the corresponding rate theories in contravariant basis, covariant basis and using Jaumann rates.

4.6 Rate constitutive theory of order one (n=1): compressible thermofluids

In this theory we limit the convected time derivative of the strain tensor to just one i.e., we only consider first convected time derivative of the strain tensor as argument of the dependent variables in the constitutive theory in addition to $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$. This constitutive theory forms the basis for generalized Newtonian and Newtonian thermofluids.

$$ \begin{aligned} [{}^{(0)}_{d}\bar{\sigma}] &= [{}^{(0)}_{d}\bar{\sigma}(\bar{\rho}\,,\, [{}^{(1)}\gamma]\,,\, \bar{\theta}\,,\, \bar{\boldsymbol{g}})] \\ {}^{(0)}\bar{\boldsymbol{q}} &= {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\,,\, [{}^{(1)}\gamma]\,,\, \bar{\theta}\,,\, \bar{\boldsymbol{g}}) \end{aligned} \quad (4.21) $$

4.6.1 Constitutive theory for the deviatoric Cauchy stress tensor

The combined generators of the argument tensors $[^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two (using table 4.1) are given by

$$
\begin{gathered}
[^{\sigma}\underset{\sim}{G}^1] = [^{(1)}\gamma] \quad ; \quad [^{\sigma}\underset{\sim}{G}^2] = [^{(1)}\gamma]^2 \quad ; \quad [^{\sigma}\underset{\sim}{G}^5] = \bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}} \\
[^{\sigma}\underset{\sim}{G}^9] = \bar{\boldsymbol{g}} \otimes [^{(1)}\gamma]\bar{\boldsymbol{g}} + [^{(1)}\gamma]\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}} \quad ; \quad [^{\sigma}\underset{\sim}{G}^{10}] = \bar{\boldsymbol{g}} \otimes [^{(1)}\gamma]^2\bar{\boldsymbol{g}} + [^{(1)}\gamma]^2\bar{\boldsymbol{g}} \otimes \bar{\boldsymbol{g}}
\end{gathered}
\tag{4.22}
$$

Let us redefine $[^{\sigma}\underset{\sim}{G}^5]$ as $[^{\sigma}\underset{\sim}{G}^3]$, $[^{\sigma}\underset{\sim}{G}^9]$ as $[^{\sigma}\underset{\sim}{G}^4]$ and $[^{\sigma}\underset{\sim}{G}^{10}]$ as $[^{\sigma}\underset{\sim}{G}^5]$. Thus the combined generators are $[^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \dots, 5$. The combined invariants of the tensors $[^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ are (using table 3.2)

$$
\begin{gathered}
{}^{q\sigma}\underset{\sim}{I}^1 = \text{tr}([^{(1)}\gamma]) \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^2 = \text{tr}([^{(1)}\gamma]^2) \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^3 = \text{tr}([^{(1)}\gamma]^3) \\
{}^{q\sigma}\underset{\sim}{I}^7 = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}} \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^{12} = \bar{\boldsymbol{g}} \cdot [^{(1)}\gamma]\,\bar{\boldsymbol{g}} \quad ; \quad {}^{q\sigma}\underset{\sim}{I}^{13} = \bar{\boldsymbol{g}} \cdot [^{(1)}\gamma]^2\,\bar{\boldsymbol{g}}
\end{gathered}
\tag{4.23}
$$

Let us redefine ${}^{q\sigma}\underset{\sim}{I}^7$ as ${}^{q\sigma}\underset{\sim}{I}^4$, ${}^{q\sigma}\underset{\sim}{I}^{12}$ as ${}^{q\sigma}\underset{\sim}{I}^5$ and ${}^{q\sigma}\underset{\sim}{I}^{13}$ as ${}^{q\sigma}\underset{\sim}{I}^6$. Thus the combined invariants are ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 6$. Therefore

$$
[^{(0)}_{\ d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + \sum_{i=1}^{5} {}^{\sigma}\alpha^i[^{\sigma}\underset{\sim}{G}^i] \tag{4.24}
$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 5$ in (4.24) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 6$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \dots, 6$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 5$ in (4.24) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 5$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 6$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{t_n} + \sum_{j=1}^{6} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n}\Big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\Big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ ; \ i = 0, 1, \dots, 5 \tag{4.25}
$$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1,2,\dots,6$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0,1,\dots,5$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,\dots,6$ whereas in (4.25), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1,2,\dots,6\ ,\ \bar{\theta}_{t_{n+1}}\ ,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1,2,\dots,6\big)$; $i = 0,1,\dots,5$. When (4.25) is substituted in (4.24), we obtain the most general first ($n = 1$) order rate constitutive theory for $[{}^{(0)}_{d}\bar{\sigma}]$ for compressible thermofluids. We follow section 4.4.3 to obtain material coefficients and the rate theories in contravariant basis, covariant basis and using Jaumann rates.

4.6.2 Constitutive theory for the heat vector

The combined generators of the argument tensors $[{}^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one are given in table 4.3.

$$\{{}^{q}\underset{\sim}{G}^1\} = \bar{\boldsymbol{g}} \qquad ; \qquad \{{}^{q}\underset{\sim}{G}^2\} = [{}^{(1)}\gamma]\,\bar{\boldsymbol{g}} \qquad ; \qquad \{{}^{q}\underset{\sim}{G}^3\} = [{}^{(1)}\gamma]^2\,\bar{\boldsymbol{g}} \tag{4.26}$$

The combined invariants remain the same as defined by (4.23). Using the combined generators (4.26) we can write

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{3} {}^{q}\alpha^i\{{}^{q}\underset{\sim}{G}^i\} \tag{4.27}$$

The coefficients ${}^{q}\alpha^i$; $i = 1,2,3$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,6$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1,2,\dots,6$. To determine the coefficients ${}^{q}\alpha^i$; $i = 1,2,3$ (in the current configuration at time $t = t_{n+1}$) in (4.27), we consider Taylor series expansion of each ${}^{q}\alpha^i$; $i = 1,2,3$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1,2,\dots,6$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{q}\alpha^i = {}^{q}\alpha^i\big|_{t_n} + \sum_{j=1}^{6} \frac{\partial({}^{q}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{q}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 1,2,3 \tag{4.28}$$

${}^{q}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{q}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1,2,\dots,6$ and $\frac{\partial({}^{q}\alpha^i)}{\partial\theta}\big|_{t_n}$; $i = 1,2,3$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,\dots,6$ whereas in (4.28), ${}^{q}\alpha^i = {}^{q}\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1,2,\dots,6, \bar{\theta}_{t_{n+1}}, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1,2,\dots,6\big)$; $i = 1,2,3$. When (4.28) is substituted in (4.27), we obtain the most general first order

$(n = 1)$ rate constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermofluids. We use remarks in section 4.4.3 to obtain material coefficients and the rate theories in the three bases.

4.7 Constitutive theory for compressible generalized Newtonian and Newtonian thermoviscous fluids

The rate constitutive theory of order one presented in section 4.6 can be modified to obtain the constitutive theory for generalized Newtonian and Newtonian fluids. We consider first order rate theory presented in section 4.6 and assume that the deviatoric Cauchy stress tensor does not depend upon $\bar{\boldsymbol{g}}$ i.e., $\bar{\boldsymbol{g}}$ is not an argument tensor of $[{}^{(0)}_{\ d}\bar{\sigma}]$, and that the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ does not depend upon $[{}^{(1)}\gamma]$ i.e., $[{}^{(1)}\gamma]$ is not an argument tensor of ${}^{(0)}\bar{\boldsymbol{q}}$.

$$[{}^{(0)}_{\ d}\bar{\sigma}] = [{}^{(0)}_{\ d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta})] \tag{4.29}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{4.30}$$

Using (4.29) and (4.30) we can derive a much simplified constitutive theory which obviously has limitations due to limiting the argument tensors in (4.29) and (4.30).

4.7.1 Constitutive theory for the deviatoric Cauchy stress tensor

In this case the generators and invariants are only due to $[{}^{(1)}\gamma]$. Thus we have

$$[{}^{\sigma}\underset{\sim}{G}^1] = [{}^{(1)}\gamma] \qquad ; \qquad [{}^{\sigma}\underset{\sim}{G}^2] = [{}^{(1)}\gamma]^2 \tag{4.31}$$

$${}^{\sigma}\underset{\sim}{I}^1 = \text{tr}([{}^{(1)}\gamma]) = i_{{}^{(1)}\gamma} \quad ; \quad {}^{\sigma}\underset{\sim}{I}^2 = \text{tr}([{}^{(1)}\gamma]^2) = ii_{{}^{(1)}\gamma} \quad ; \quad {}^{\sigma}\underset{\sim}{I}^3 = \text{tr}([{}^{(1)}\gamma]^3) = iii_{{}^{(1)}\gamma} \tag{4.32}$$

$$[{}^{(0)}_{\ d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{\sigma}\underset{\sim}{G}^1] + {}^{\sigma}\alpha^2[{}^{\sigma}\underset{\sim}{G}^2] \tag{4.33}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ in (4.33) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1,2,3$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1,2,3$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0,1,2$ in (4.33) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0,1,2$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1,2,3$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{3} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 0,1,2 \tag{4.34} $$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1,2,3$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0,1,2$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,3$ whereas in (4.34), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1,2,3\,,\,\bar{\theta}_{t_{n+1}}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1,2,3\big)$; $i = 0,1,2$. If we let $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)} = {}^{\sigma}\alpha^i{}_{,j}$; $j = 1,2,3$ and $i = 0,1,2$ then (4.34) can be written as

$$ {}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{3} ({}^{\sigma}\alpha^i{}_{,j})_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \quad ; \quad i = 0,1,2 \tag{4.35} $$

Substituting from (4.32) into (4.35) and then (4.35) into (4.33)

$$ \begin{aligned} [{}^{(0)}_{d}\bar{\sigma}] =& \Big({}^{\sigma}\alpha^0\big|_{t_n} + ({}^{\sigma}\alpha^0{}_{,1})_{t_n}\big((i_{{}^{(1)}\gamma})_{t_{n+1}} - (i_{{}^{(1)}\gamma})_{t_n}\big) + ({}^{\sigma}\alpha^0{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\ &+ ({}^{\sigma}\alpha^0{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] \qquad + \\ &\Big({}^{\sigma}\alpha^1\big|_{t_n} + ({}^{\sigma}\alpha^1{}_{,1})_{t_n}\big((i_{{}^{(1)}\gamma})_{t_{n+1}} - (i_{{}^{(1)}\gamma})_{t_n}\big) + ({}^{\sigma}\alpha^1{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\ &+ ({}^{\sigma}\alpha^1{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma] \quad + \\ &\Big({}^{\sigma}\alpha^2\big|_{t_n} + ({}^{\sigma}\alpha^2{}_{,1})_{t_n}\big((i_{{}^{(1)}\gamma})_{t_{n+1}} - (i_{{}^{(1)}\gamma})_{t_n}\big) + ({}^{\sigma}\alpha^2{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\ &+ ({}^{\sigma}\alpha^2{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma]^2 \end{aligned} \tag{4.36} $$

In (4.36), we note that all quantities at time t_n are known as these correspond to a configuration for which the deformation field is known. We collect terms in (4.36) and define material coefficients and others. Let

$$
\begin{aligned}
&\bar{\sigma}_0\big|_{t_n} = {}^{\sigma}\alpha^0\big|_{t_n} - ({}^{\sigma}\alpha^0{}_{,1})_{t_n}(i_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^0{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^0{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_1 = ({}^{\sigma}\alpha^0{}_{,1})_{t_n} \quad ; \quad {}^{\sigma}b_2 = ({}^{\sigma}\alpha^0{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_3 = ({}^{\sigma}\alpha^0{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^1 = {}^{\sigma}\alpha^1\big|_{t_n} - ({}^{\sigma}\alpha^1{}_{,1})_{t_n}(i_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^1{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^1{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_2^1 = ({}^{\sigma}\alpha^1{}_{,1})_{t_n} \quad ; \quad {}^{\sigma}b_3^1 = ({}^{\sigma}\alpha^1{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_4^1 = ({}^{\sigma}\alpha^1{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^2 = {}^{\sigma}\alpha^2\big|_{t_n} - ({}^{\sigma}\alpha^2{}_{,1})_{t_n}(i_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^2{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^2{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_2^2 = ({}^{\sigma}\alpha^2{}_{,1})_{t_n} \quad ; \quad {}^{\sigma}b_3^2 = ({}^{\sigma}\alpha^2{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_4^2 = ({}^{\sigma}\alpha^2{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^3 = \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^{\sigma}b_2^3 = \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^{\sigma}b_3^3 = \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n}
\end{aligned}
\tag{4.37}
$$

Using (4.37) we can write (4.36) as

$$
\begin{aligned}
[{}^{(0)}_{\,d}\bar{\sigma}] = {}& \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_1(i_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] \\
& + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_2^1(i_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] + {}^{\sigma}b_3^1(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] + {}^{\sigma}b_4^1(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] \\
& + {}^{\sigma}b_1^2[{}^{(1)}\gamma]^2 + {}^{\sigma}b_2^2(i_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 + {}^{\sigma}b_3^2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 + {}^{\sigma}b_4^2(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 \\
& + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] + {}^{\sigma}b_2^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma] + {}^{\sigma}b_3^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma]^2
\end{aligned}
\tag{4.38}
$$

in which $\bar{\sigma}_0\big|_{t_n}$, ${}^{\sigma}b_j$; $j = 1,2,3$, ${}^{\sigma}b_i^1$, ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_k^3$; $k = 1,2,3$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1,2,3$ as evident from (4.37). $\bar{\sigma}_0\big|_{t_n}$ is the stress field associated with the configuration at time $t = t_n$. ${}^{\sigma}b_j$; $j = 1,2,3$, ${}^{\sigma}b_i^1$, ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_k^3$; $k = 1,2,3$ are variable material coefficients that are deformation dependent during the evolution. (4.38) are the constitutive equations for $[{}^{(0)}_{\,d}\bar{\sigma}]$ for *compressible generalized Newtonian thermofluids with variable material coefficients*. This theory requires determination of variable material coefficients ${}^{\sigma}b_j$; $j = 1,2,3$, ${}^{\sigma}b_i^1$, ${}^{\sigma}b_i^2$; $i = 1,2,\dots,4$ and ${}^{\sigma}b_k^3$; $k = 1,2,3$, a total of fourteen.

(a) Further assumptions and simplifications

The constitutive theory described by (4.38) can be further simplified if we make the following assumptions:

(i) We neglect the generators $[{}^{(1)}\gamma]^2$ all together in the development of the rate constitutive theory. Thus the terms in (4.38) containing the coefficients ${}^{\sigma}b_i^2$; $i = 1, 2, 3$ and ${}^{\sigma}b_3^3$ can be deleted.

(ii) In (4.38), we neglect all product terms in the current configuration $t = t_{n+1}$ i.e., the products of generator $[{}^{(1)}\gamma]$ and its invariants can be deleted from (4.38) including the products of $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ with $[{}^{(1)}\gamma]$.

(iii) When using simplifications (i) and (ii), we ensure that the material coefficients defined in (4.37) are not affected. With these assumptions, (4.38) reduces to

$$
\begin{aligned}
[{}^{(0)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_1(i_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] \\
+ {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{4.39}
$$

(4.39) is a much simplified constitutive theory for $[{}^{(0)}_{\ d}\bar{\sigma}]$. It only requires determination of five (${}^{\sigma}b_1$, ${}^{\sigma}b_2$, ${}^{\sigma}b_3$, ${}^{\sigma}b_1^1$, ${}^{\sigma}b_1^3$) material coefficients.

(iv) If we further assume the constitutive theory for $[{}^{(0)}_{\ d}\bar{\sigma}]$ to be linear in the components of $[{}^{(1)}\gamma]$, then $(ii_{{}^{(1)}\gamma})_{t_{n+1}}$ and $(iii_{{}^{(1)}\gamma})_{t_{n+1}}$ terms in (4.39) can be deleted.

$$
[{}^{(0)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_1(i_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\tag{4.40}
$$

We redefine the material coefficients to conform to commonly used notations.

$$
\begin{aligned}
{}^{\sigma}b_1 &= \kappa\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,(i_{{}^{(1)}\gamma})_{t_n}\,,\,(ii_{{}^{(1)}\gamma})_{t_n}\,,\,(iii_{{}^{(1)}\gamma})_{t_n}\big) = \kappa_{t_n} \\
{}^{\sigma}b_1^1 &= 2\eta\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,(i_{{}^{(1)}\gamma})_{t_n}\,,\,(ii_{{}^{(1)}\gamma})_{t_n}\,,\,(iii_{{}^{(1)}\gamma})_{t_n}\big) = 2\eta_{t_n} \\
{}^{\sigma}b_1^3 &= -\alpha_{tm}\big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,(i_{{}^{(1)}\gamma})_{t_n}\,,\,(ii_{{}^{(1)}\gamma})_{t_n}\,,\,(iii_{{}^{(1)}\gamma})_{t_n}\big) = -(\alpha_{tm})_{t_n}
\end{aligned}
\tag{4.41}
$$

$$\therefore \qquad [{}^{(0)}_{\ d}\bar{\sigma}] = \left.\bar{\sigma}_0\right|_{t_n}[I] + \kappa_{t_n}\mathrm{tr}([{}^{(1)}\gamma])[I] + 2\eta_{t_n}[{}^{(1)}\gamma] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.42}$$

(4.42) is the most simplified constitutive theory for the deviatoric Cauchy stress tensor for *compressible generalized Newtonian thermofluids with variable material coefficients*. This constitutive model for $[{}^{(0)}_{\ d}\bar{\sigma}]$ only requires determination of three material coefficients.

Remarks:

1. We note that $[{}^{(0)}_{\ d}\bar{\sigma}]$, $\mathrm{tr}([{}^{(1)}\gamma])$ and $[{}^{(1)}\gamma]$ in (4.42) are defined in the configuration at time $t = t_{n+1}$ but the coefficients κ_{t_n}, η_{t_n} and $(\alpha_{tm})_{t_n}$ are obviously defined in the configuration at time $t = t_n$.

2. η and κ are *first* and *second viscosities*, and α_{tm} is the *thermal modulus*.

3. By replacing $\big([{}^{(0)}_{\ d}\bar{\sigma}], \mathrm{tr}([{}^{(1)}\gamma])$ and $[{}^{(1)}\gamma]\big)$ with $\big([{}_d\bar{\sigma}^{(0)}], \mathrm{tr}([\gamma^{(1)}])$ and $[\gamma^{(1)}]\big)$, $\big([{}_d\bar{\sigma}_{(0)}], \mathrm{tr}([\gamma_{(1)}])$ and $[\gamma_{(1)}]\big)$ and $\big([{}^{(0)}_{\ d}\bar{\sigma}^J], \mathrm{tr}([{}^{(1)}\gamma^J])$ and $[{}^{(1)}\gamma^J]\big)$ in (4.42), we obtain forms of (4.42) in contravariant basis, covariant basis and in Jaumann rates. Keeping in mind that the arguments of η, κ and α_{tm} that are dependent on $[{}^{(1)}\gamma]$ are likewise modified.We note that

$$[{}^{(1)}\gamma] = [\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}] \tag{4.43}$$

Hence, right side of (4.42) remains unaffected by the choice of the basis, thus $[{}^{(0)}_{\ d}\bar{\sigma}] = [{}_d\bar{\sigma}^{(0)}] = [{}_d\bar{\sigma}_{(0)}] = [{}^{(0)}_{\ d}\bar{\sigma}^J] = [{}_d\bar{\sigma}]$ holds and we can write

$$[{}_d\bar{\sigma}] = \left.\bar{\sigma}_0\right|_{t_n}[I] + \kappa_{t_n}\mathrm{tr}([\bar{D}])[I] + 2\eta_{t_n}[\bar{D}] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.44}$$

(b) Variable transport properties or material coefficients

From (4.41), we note that η, κ and α_{tm} can be functions of density and temperature during the evolution but their values must be determined based on $\bar{\rho}$ and $\bar{\theta}$ in the immediately preceding known configuration ($t = t_n$ in this case). (4.41) permit us to use experimentally and/or empirically determined relations for density and temperature dependent η, κ and α_{tm}. For example

$$\eta_{t_n} = \eta^0 \Big(\frac{\bar{\theta}_{t_n}}{\theta^0}\Big)^{\underset{\sim}{n}} \qquad ; \qquad \text{Power law} \tag{4.45}$$

$$\eta_{t_n} = \eta^0 \Big(\frac{\bar{\theta}_{t_n}}{\theta^0}\Big)^{3/2} \Big(\frac{\theta^0 + s}{\bar{\theta}_{t_n} + s}\Big) \qquad ; \qquad \text{Sutherland law} \tag{4.46}$$

are valid for viscosity η. The parameters η^0, θ^0, s and $\underset{\sim}{n}$ are known constants for a given fluid. Similar relations could also be used for κ_{t_n} and $(\alpha_{tm})_{t_n}$. Dependence of η, κ and α_{tm} on $\bar{\rho}_{t_n}$ can also be considered in a similar fashion. This permits us to incorporate variable transport properties η, κ and α_{tm} dependent on density and temperature during the evolution with the exception that η, κ and α_{tm} must be determined in the configuration at $t = t_n$ whereas $[{}_d\bar{\sigma}]$ and $[\bar{D}]$ in (4.44) are for the current configuration at time $t = t_{n+1}$ i.e., the constitutive equations (4.44) hold for the current configuration at $t = t_{n+1}$.

Power law, Carreau-Yasuda etc. models for η and κ

From (4.41), we note that η and κ can be functions of $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$ also. This allows us to define η and κ as functions of $i_{\bar{D}}$, $ii_{\bar{D}}$ and $iii_{\bar{D}}$ using experimentally and/or empirically determined relations. Power law and Carreau-Yasuda models for shear thinning and shear thickening fluids are examples and are obviously justified based on (4.41). Thus, power law and Carreau-Yasuda models in fact have continuum mechanics basis. To relate to the published work in polymer science for these models, we consider the following.

Power law [22]: Let

$$[\dot{\gamma}] = [\bar{L}] + [\bar{L}]^T \tag{4.47}$$

$$ii_{\dot{\gamma}} = \text{tr}([\dot{\gamma}]^2) \tag{4.48}$$

a scalar $\dot{\gamma}$ is defined as

$$\overset{\bullet}{\gamma} = \sqrt{\frac{1}{2} ii_{\overset{\bullet}{\gamma}}} \tag{4.49}$$

then we express η as a function of $\overset{\bullet}{\gamma}$

$$\eta = m(\overset{\bullet}{\gamma})^{\tilde{n}-1} \tag{4.50}$$

We conduct experiments to determine a graph of η versus $\overset{\bullet}{\gamma}$ (on log-log scale) from which the power law index $\tilde{n}$ is determined for a specific fluid of interest. We note that

$$[\overset{\bullet}{\gamma}] = 2[\bar{D}] \tag{4.51}$$

Hence

$$ii_{\overset{\bullet}{\gamma}} = \text{tr}([\overset{\bullet}{\gamma}]^2) = 4\text{tr}([\bar{D}]^2) = 4ii_{\bar{D}} \tag{4.52}$$

$$\therefore \qquad \overset{\bullet}{\gamma} = \sqrt{\frac{1}{2} ii_{\overset{\bullet}{\gamma}}} = \sqrt{\frac{1}{2}(4ii_{\bar{D}})} = \sqrt{2ii_{\bar{D}}} \tag{4.53}$$

Substituting from (4.53) into (4.50)

$$\eta = m(2ii_{\bar{D}})^{\frac{\tilde{n}-1}{2}} \tag{4.54}$$

more specifically, we write

$$\eta_{t_n} = m(2(ii_{\bar{D}})_{t_n})^{\frac{\tilde{n}-1}{2}} = \eta\big((ii_{\bar{D}})_{t_n}\big) \tag{4.55}$$

In (4.55), we have considered η_{t_n} as a function of $(ii_{\bar{D}})_{t_n}$ only and not a function of $(i_{\bar{D}})_{t_n}$ and/or $(iii_{\bar{D}})_{t_n}$. m is a constant for the fluid. For compressible thermofluids, dependence of η_{t_n} on $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$ is valid based on the derivation presented here.

***Carreau-Yasuda [22]*:** Based on (4.41), we can also consider any other experimental and/or empirical relationship expressing η_{t_n} as a function of $(ii_{\bar{D}})_{t_n}$. Carreau-Yasuda model is an example.

$$\eta_{t_n} = n^{\infty} + (n^0 - n^{\infty})\big(1 + (\lambda\sqrt{2(ii_{\bar{D}})_{t_n}}\,)^a\big)^{\frac{\tilde{n}-1}{a}} = \eta\big((ii_{\bar{D}})_{t_n}\big) \tag{4.56}$$

n^0, n^{∞}, a, λ and $\tilde{n}$ are model constants [22]. This model obviously also has continuum mechanics basis based on (4.41). In fact, any desired experimental and/or empirical relationship expressing η_{t_n} as a function of $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$ is permissible based on (4.41).

***Remarks*:**

1. When η, κ and α_{tm} in (4.44) show dependence on $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$, $(i_{\bar{D}})_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$ as in (4.41), we refer to the fluid described by (4.44) as *compressible generalized Newtonian thermoviscous fluid with variable transport properties*. Thus, Power law and Carreau-Yasuda models are special cases of a more broader category of fluids described by (4.41) and (4.44).

2. When η, κ and α_{tm} in (4.44) only show dependence on $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ i.e., when

$$\begin{aligned} \eta_{t_n} &= \eta(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}) \\ \kappa_{t_n} &= \kappa(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}) \end{aligned} \tag{4.57}$$

   Then, (4.44) and (4.57) describe *compressible Newtonian thermoviscous fluids with variable transport properties*.

3. The first term in (4.44) is due to the initial stress field in the configuration at time t_n and the last term accounts for the stress field created in the current configuration due to the expansion or contraction compared to the configuration at time $t = t_n$.

4. It is important to emphasize that the constitutive equations such as (4.44) hold for the current configuration at $t = t_{n+1}$. Thus in (4.44) $[{}_d\bar{\sigma}]$, $[\bar{D}]$ are in the current configuration at $t = t_{n+1}$. However η_{t_n}, κ_{t_n} and $(\alpha_{tm})_{t_n}$ are determined based on the known deformation field in the configuration corresponding to $t = t_n$. This is a consequence of the Taylor series

expansion about the configuration at time $t = t_n$ of the coefficients ${}^{\sigma}\alpha^i$ in (4.33). In the current published works ([22, 38, 39]), this is not the case, but instead η, κ and α_{tm} are treated as the functions of unknown deformation field in the current configuration at time $t = t_{n+1}$ i.e., instead of η_{t_n}, κ_{t_n} and $(\alpha_{tm})_{t_n}$, these are replaced by

$$
\begin{aligned}
\eta_{t_{n+1}} &= \eta(\bar{\rho}_{t_{n+1}}\, ,\, \bar{\theta}_{t_{n+1}}\, ,\, (i_{\bar{D}})_{t_{n+1}}\, ,\, (ii_{\bar{D}})_{t_{n+1}}\, ,\, (iii_{\bar{D}})_{t_{n+1}}) = \eta \\
\kappa_{t_{n+1}} &= \kappa(\bar{\rho}_{t_{n+1}}\, ,\, \bar{\theta}_{t_{n+1}}\, ,\, (i_{\bar{D}})_{t_{n+1}}\, ,\, (ii_{\bar{D}})_{t_{n+1}}\, ,\, (iii_{\bar{D}})_{t_{n+1}}) = \kappa \\
(\alpha_{tm})_{t_{n+1}} &= \alpha_{tm}(\bar{\rho}_{t_{n+1}}\, ,\, \bar{\theta}_{t_{n+1}}\, ,\, (i_{\bar{D}})_{t_{n+1}}\, ,\, (ii_{\bar{D}})_{t_{n+1}}\, ,\, (iii_{\bar{D}})_{t_{n+1}}) = \alpha_{tm}
\end{aligned}
\tag{4.58}
$$

In (4.58), we have redefined $\eta_{t_{n+1}}$, $\kappa_{t_{n+1}}$ and $(\alpha_{tm})_{t_{n+1}}$ by η, κ and α_{tm} (their values in the current configuration). With the new definitions of η, κ and α_{tm} in (4.58), (4.44) can be written as

$$
[{}_d\bar{\sigma}] = \left. \bar{\sigma}_0 \right|_{t_n} [I] + \kappa \mathrm{tr}([\bar{D}])[I] + 2\eta[\bar{D}] - \alpha_{tm}(\bar{\theta} - \bar{\theta}_{t_n})[I] \tag{4.59}
$$

With the new definitions in (4.58), power law, Sutherland law for temperature dependent viscosity, and power and Carreau-Yasuda models (equations (4.45), (4.46) and (4.55), (4.56)) can be written as

$$
\eta_{t_{n+1}} = \eta = \eta^0 \Big(\frac{\bar{\theta}}{\theta^0}\Big)^{\underset{\sim}{n}} \quad ; \quad \text{Power law: temperature dependent viscosity} \tag{4.60}
$$

$$
\eta_{t_{n+1}} = \eta = \eta^0 \Big(\frac{\bar{\theta}}{\theta^0}\Big)^{3/2} \Big(\frac{\theta^0 + s}{\bar{\theta} + s}\Big) \ ; \ \text{Sutherland law: temperature dependent viscosity} \tag{4.61}
$$

$$
\eta_{t_{n+1}} = \eta = m(2 ii_{\bar{D}})^{\frac{\tilde{n}-1}{2}} \quad ; \quad \text{Power law: Generalized Newtonian fluids} \tag{4.62}
$$

$$
\eta_{t_{n+1}} = \eta = n^{\infty} + (n^0 - n^{\infty}) \big(1 + (\lambda \sqrt{2 ii_{\bar{D}}}\,)^a\big)^{\frac{\tilde{n}-1}{a}} \ ; \ \begin{array}{c}\text{Carreau-Yasuda model:} \\ \text{Generalized Newtonian fluids}\end{array} \tag{4.63}
$$

(4.59) with (4.60) - (4.63) are what is used currently in the published works. When the two configurations at time t_n and t_{n+1} are in close proximity in terms of deformation field, using (4.58) may be justified but it is not supported by the derivations presented here.

4.7.2 Constitutive theory for the heat vector

Based on (4.30), in this case the only generator is $\bar{\boldsymbol{g}}$, hence we can write

$$ {}^{(0)}\bar{\boldsymbol{q}} = -{}^{q}\alpha\, \bar{\boldsymbol{g}} \tag{4.64}$$

Also, the only invariant is ${}^{q}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$, hence the coefficient ${}^{q}\alpha$ in (4.64) is a function of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q}\underset{\sim}{I})_{t_{n+1}}$. To determine the coefficient ${}^{q}\alpha$ in (4.64) related to the current configuration at time $t = t_{n+1}$, we consider Taylor series expansion of ${}^{q}\alpha$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}$ and retain only up to linear terms in $\bar{\theta}$ and the invariant.

$$ {}^{q}\alpha = {}^{q}\alpha\big|_{t_n} + \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\bigg|_{t_n} \big(({}^{q}\underset{\sim}{I})_{t_{n+1}} - ({}^{q}\underset{\sim}{I})_{t_n} \big) + \frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \tag{4.65}$$

${}^{q}\alpha\big|_{t_n}$, $\frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\big|_{t_n}$ and $\frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\big|_{t_n}$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q}\underset{\sim}{I})_{t_n}$ whereas from equation (4.65) we have ${}^{q}\alpha = {}^{q}\alpha\big(\bar{\rho}_{t_n}\,,\ \bar{\theta}_{t_n}\,,\ ({}^{q}\underset{\sim}{I})_{t_n}\,,\ \bar{\theta}_{t_{n+1}}\,,\ ({}^{q}\underset{\sim}{I})_{t_{n+1}}\big)$. Substituting from (4.65) into (4.64)

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\Big({}^{q}\alpha\big|_{t_n} + \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\bigg|_{t_n} \big(({}^{q}\underset{\sim}{I})_{t_{n+1}} - ({}^{q}\underset{\sim}{I})_{t_n} \big) + \frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \Big)\, \bar{\boldsymbol{g}} \tag{4.66}$$

or
$$ {}^{(0)}\bar{\boldsymbol{q}} = -{}^{q}\alpha\big|_{t_n} \bar{\boldsymbol{g}} - \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\bigg|_{t_n} \big((\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_{n+1}} - (\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_n} \big)\, \bar{\boldsymbol{g}} - \frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\, \bar{\boldsymbol{g}} \tag{4.67}$$

We note that if there is a uniform temperature change between the configurations at times t_n and t_{n+1}, then $\bar{\boldsymbol{g}} = 0$ and hence ${}^{(0)}\bar{\boldsymbol{q}}$ must be zero. This condition is satisfied by (4.67). We note that all quantities at time t_n are known as these correspond to a configuration for which the deformation field is known. We collect terms and define

$$ k_{t_n} = {}^{q}\alpha\big|_{t_n} - \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\bigg|_{t_n} (\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_n} \quad ; \quad (k_1)_{t_n} = \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\bigg|_{t_n} \quad ; \quad (k_2)_{t_n} = \frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\bigg|_{t_n} \tag{4.68}$$

then (4.67) becomes (we drop the subscript t_{n+1} since it is understood it represents the current configuration)

$$ {}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} - (k_1)_{t_n}(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})\,\bar{\boldsymbol{g}} - (k_2)_{t_n}(\bar{\theta}-\bar{\theta}_{t_n})\,\bar{\boldsymbol{g}} \tag{4.69} $$

We note that $k_{t_n} = k(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$, $(k_1)_{t_n} = k_1(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$ and $(k_2)_{t_n} = k_2(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$. (4.69) is the most general form of the constitutive equation for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ based on (4.30). If we neglect If we neglect the last term in (4.69) then

$$ {}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} - (k_1)_{t_n}(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})\,\bar{\boldsymbol{g}} \tag{4.70} $$

and if we neglect infinitesimals of order two and higher in the components of $\bar{\boldsymbol{g}}$, then

$$ {}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} \tag{4.71} $$

$$ \text{or} \qquad {}^{(0)}\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} = -k[I]\,\bar{\boldsymbol{g}} = -[K]\,\bar{\boldsymbol{g}} \tag{4.72} $$

in which k is *thermal conductivity* and $[K]$ is the *diagonal thermal conductivity matrix*. We note that

$$ k = k_{t_n} = k\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}\big) \tag{4.73} $$

Based on 4.73, the thermal conductivity can be a function of density, temperature and the first invariant of $\bar{\boldsymbol{g}}$ i.e., $\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}}$ in the configuration at time $t = t_n$. Thus, as in case of viscosity, here also we can use experimental and/or empirical relation for thermal conductivity as a function of $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}}$, keeping in mind that (4.69) and (4.72) hold for the current configuration at time $t = t_{n+1}$ whereas material coefficients in (4.69) and (4.72) are only defined for the configuration at time $t = t_n$. Thus, power law, Sutherland law for temperature dependent k [22, 38, 39] are justified.

$$ k_{t_n} = k^0\Big(\frac{\bar{\theta}_{t_n}}{\theta^0}\Big)^{\underline{n}} \qquad ; \qquad \text{Power law} \tag{4.74} $$

$$ k_{t_n} = k^0\Big(\frac{\bar{\theta}_{t_n}}{\theta^0}\Big)^{3/2}\Big(\frac{\theta^0+\underline{s}}{\bar{\theta}_{t_n}+\underline{s}}\Big) \qquad ; \qquad \text{Sutherland law} \tag{4.75} $$

k^0, θ^0, $\underline{n}$ and $\underline{s}$ are constants for a specific fluid. This permits us to have variable thermal conductivity during the evolution. Similarly we can also consider k as a function of $\bar{\rho}_{t_n}$ as well, keeping in mind that based on 4.73, dependence of k_{t_n} on $(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_n}$ is permissible as well.

***Remarks*:**

1. The constitutive theory for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ based on the generator of the argument tensor $\bar{\boldsymbol{g}}$ given by (4.69) is much more complicated than the standard Fourier heat conduction law defined by (4.72) which only considers first degree terms in the components of $\bar{\boldsymbol{g}}$ in the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$.

2. In (4.69) as well as (4.72), $\bar{\boldsymbol{g}}$ is independent of the basis, hence (4.69) and (4.72) are valid in contra- and co-variant basis as well as Jaumann i.e.

$$ {}^{(0)}\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}_{(0)} = {}^{(0)}\bar{\boldsymbol{q}}^{J} = \bar{\boldsymbol{q}} \tag{4.76} $$

3. The constitutive equations (4.69) and (4.72) hold for the current configuration at time $t = t_{n+1}$, thus ${}^{(0)}\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ correspond to time $t = t_{n+1}$, however, the coefficients k, ${}^{1}k$ and ${}^{2}k$ are defined in the configuration at time $t = t_n$. This is obviously a consequence of the Taylor series expansion of ${}^{q}\alpha$ about the configuration at $t = t_n$. In the currently published works [22, 38, 39] this is not the case, but instead k, ${}^{1}k$ and ${}^{2}k$ are treated as functions of the unknown field in the current configuration, that is k_{t_n}, ${}^{1}k_{t_n}$ and ${}^{2}k_{t_n}$ are replaced by

$$ \begin{aligned} k_{t_{n+1}} &= k\big(\bar{\rho}_{t_{n+1}}\,,\,\bar{\theta}_{t_{n+1}}\,,\,(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_{n+1}}\big) = k\big(\bar{\rho}\,,\,\bar{\theta}\,,\,\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}\big) \\ {}^{1}k_{t_{n+1}} &= {}^{1}k\big(\bar{\rho}_{t_{n+1}}\,,\,\bar{\theta}_{t_{n+1}}\,,\,(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_{n+1}}\big) = {}^{1}k\big(\bar{\rho}\,,\,\bar{\theta}\,,\,\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}\big) \\ {}^{2}k_{t_{n+1}} &= {}^{2}k\big(\bar{\rho}_{t_{n+1}}\,,\,\bar{\theta}_{t_{n+1}}\,,\,(\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_{n+1}}\big) = {}^{2}k\big(\bar{\rho}\,,\,\bar{\theta}\,,\,\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}\big) \end{aligned} \tag{4.77} $$

   In (4.77), we have redefined k, ${}^{1}k$ and ${}^{2}k$ by $k_{t_{n+1}}$, ${}^{1}k_{t_{n+1}}$ and ${}^{2}k_{t_{n+1}}$ in the current configuration at time $t = t_{n+1}$. With the new definitions (4.77), (4.69) and (4.72) can be written as

$$\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} - {}^{1}k\big((\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}} - (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}\big)\,\bar{\boldsymbol{g}} - {}^{2}k(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\,\bar{\boldsymbol{g}} \tag{4.78}$$

$$\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} \tag{4.79}$$

The power law and Sutherland law ((4.74) and (4.75)) are accordingly modified

$$k = k^0\Big(\frac{\bar{\theta}}{\theta^0}\Big)^{\underline{n}} \quad ; \quad \text{Power law} \tag{4.80}$$

$$k = k^0\Big(\frac{\bar{\theta}}{\theta^0}\Big)^{3/2}\Big(\frac{\theta^0+\underline{s}}{\bar{\theta}+\underline{s}}\Big) \quad ; \quad \text{Sutherland law} \tag{4.81}$$

(4.79) - (4.81) are what is used in the published works. When the two configurations at times t_n and t_{n+1} are in close proximity of each other in terms of the deformation field, using (4.77) may be justified, but it is not supported by the derivation presented here.

4.8 Incompressible ordered thermofluids of orders *n*, 2, 1

All derivations presented so far for the constitutive theories of orders n, 2 and 1 for $[{}^{(0)}_{\;d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ assumed the thermofluid to be compressible. In this section we consider ordered rate constitutive theories for $[{}^{(0)}_{\;d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ for incompressible thermofluids. For incompressible matter

$$\bar{\rho} = \rho_{_0} = \text{constant} \tag{4.82}$$

$$\text{div}(\bar{\boldsymbol{v}}) = 0 \tag{4.83}$$

$$\therefore \quad \text{tr}([{}^{(1)}\gamma]) = \text{tr}([\gamma^{(1)}]) = \text{tr}([\gamma_{(1)}]) = \text{tr}([{}^{(1)}\gamma^{J}]) = \text{tr}([\bar{D}]) = 0 \tag{4.84}$$

$$\det([J]) = 1 \tag{4.85}$$

Thus, the density $\bar{\rho}$ can be eliminated from the argument tensors of the dependent variables $[{}^{(0)}_{\;d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ in the rate constitutive theory for incompressible thermofluids.

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\theta}(\bar{\boldsymbol{x}}, t)\big) \tag{4.86}$$

$$[{}^{(0)}\bar{\sigma}\,] = \big[{}^{(0)}_{\ e}\bar{\sigma}\big(\bar{\theta}(\bar{\boldsymbol{x}}, t)\big)\big] + [{}^{(0)}_{\ d}\bar{\sigma}\,] \tag{4.87}$$

$$[{}^{(0)}_{\ d}\bar{\sigma}\,] = \big[{}^{(0)}_{\ d}\bar{\sigma}\big([{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)]\,;\, j = 1, 2, \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t)\big)\big] \tag{4.88}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\,[{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)]\,;\, j = 1, 2 \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t)\big) \tag{4.89}$$

$$[{}^{(0)}_{\ e}\bar{\sigma}] = \bar{p}(\bar{\theta}(\bar{\boldsymbol{x}}, t))[I] \tag{4.90}$$

The rate constitutive theories of orders n, 2 and 1 for compressible thermofluids presented in earlier sections can be modified by: (i) eliminating $\bar{\rho}$ all together from the entire derivations and (ii) incorporating incompressibility conditions using (4.83) - (4.85) to obtain rate constitutive theories of orders n, 2 and 1 for incompressible thermofluids. Details are straight forward, hence not presented here for the sake of brevity. By replacing $[{}^{(0)}_{\ d}\bar{\sigma}]$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]\ ;\, j = 1, 2, \dots, n$ with the appropriate corresponding measures in the chosen basis ,i.e., $\big([{}_d\bar{\sigma}^{(0)}], \bar{\boldsymbol{q}}^{(0)}, [\gamma^{(j)}]\ ;\, j = 1, 2, \dots, n\big)$, $\big([{}_d\bar{\sigma}_{(0)}], \bar{\boldsymbol{q}}_{(0)}, [\gamma_{(j)}]\ ;\, j = 1, 2, \dots, n\big)$ and $\big([{}^{(0)}_{\ d}\bar{\sigma}^J], {}^{(0)}\bar{\boldsymbol{q}}^J, [{}^{(j)}\gamma^J]\ ;\, j = 1, 2, \dots, n\big)$ we can easily obtain the rate theories of orders n, 2 and 1 in contravariant basis, covariant basis and Jaumann basis.

4.9 Constitutive theories for incompressible generalized Newtonian and Newtonian fluids

Following the rate constitutive theories for compressible generalized Newtonian and Newtonian fluids presented in section 4.7 and eliminating $\bar{\rho}$ from the argument tensors, (4.29) - (4.30) reduces to

$$[{}^{(0)}_{\ d}\bar{\sigma}] = [{}^{(0)}_{\ d}\bar{\sigma}([{}^{(1)}\gamma]\ ,\ \bar{\theta})] \tag{4.91}$$

$$ {}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{4.92}$$

4.9.1 Constitutive theory for the deviatoric Cauchy stress tensor

(4.91) and (4.92) are exactly same as (4.29) and (4.30) used for compressible case, except that $\bar{\rho}$ is eliminated in (4.91) and (4.92). Thus the derivation follows the compressible case except that $\bar{\rho}$ is eliminated in the derivation. Generators $[{}^{\sigma}\underset{\sim}{G}^1] = [{}^{(1)}\gamma]$ and $[{}^{\sigma}\underset{\sim}{G}^2] = [{}^{(1)}\gamma]^2$ of the argument tensor $[{}^{(1)}\gamma]$ allow us to represent $[{}^{(0)}_{\ d}\bar{\sigma}]$

$$[{}^{(0)}_{\ d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{(1)}\gamma] + {}^{\sigma}\alpha^2[{}^{(1)}\gamma]^2 \tag{4.93}$$

The invariants of $[{}^{(1)}\gamma]$ are

$${}^{q}\underset{\sim}{I}^1 = i_{{}^{(1)}\gamma} = \mathrm{tr}([{}^{(1)}\gamma]) = 0 \quad ; \quad {}^{q}\underset{\sim}{I}^2 = ii_{{}^{(1)}\gamma} = \mathrm{tr}([{}^{(1)}\gamma]^2) \quad ; \quad {}^{q}\underset{\sim}{I}^3 = iii_{{}^{(1)}\gamma} = \mathrm{tr}([{}^{(1)}\gamma]^3) \tag{4.94}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0,1,2$ in (4.93) are functions of temperature $\bar{\theta}$ and the invariants ${}^{\sigma}\underset{\sim}{I}^2$ and ${}^{\sigma}\underset{\sim}{I}^3$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\theta}_{t_{n+1}}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 2,3$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0,1,2$ in (4.93) related to the configuration at time $t = t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0,1,2$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 2,3$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$\begin{aligned} {}^{\sigma}\alpha^i = \ & {}^{\sigma}\alpha^i\Big|_{t_n} + \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^2)}\bigg|_{t_n}\Big(({}^{\sigma}\underset{\sim}{I}^2)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^2)_{t_n}\Big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^3)}\bigg|_{t_n}\Big(({}^{\sigma}\underset{\sim}{I}^3)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^3)_{t_n}\Big) \\ & + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \qquad ; \qquad i = 0,1,2 \end{aligned} \tag{4.95}$$

Let us introduce the notation $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)} = {}^{\sigma}\alpha^i{}_{,j}$; $j = 2,3$ and $i = 0,1,2$. Substituting from (4.94) into (4.95) and then (4.95) into (4.93)

$$
\begin{aligned}
[{}^{(0)}_{d}\bar{\sigma}] =& \Big({}^{\sigma}\alpha^{0}\big|_{t_n} + ({}^{\sigma}\alpha^{0}{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
&+ ({}^{\sigma}\alpha^{0}{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{0})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] \; + \\
&\Big({}^{\sigma}\alpha^{1}\big|_{t_n} + ({}^{\sigma}\alpha^{1}{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
&+ ({}^{\sigma}\alpha^{1}{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{1})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma] \; + \\
&\Big({}^{\sigma}\alpha^{2}\big|_{t_n} + ({}^{\sigma}\alpha^{2}{}_{,2})_{t_n}\big((ii_{{}^{(1)}\gamma})_{t_{n+1}} - (ii_{{}^{(1)}\gamma})_{t_n}\big) \\
&+ ({}^{\sigma}\alpha^{2}{}_{,3})_{t_n}\big((iii_{{}^{(1)}\gamma})_{t_{n+1}} - (iii_{{}^{(1)}\gamma})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{2})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma]^{2}
\end{aligned}
\tag{4.96}
$$

In (4.96), we note that all quantities at time t_n are known as they correspond to a configuration (t_n) for which the deformation field is known. We collect terms in (4.96) and define material coefficients and others. Let

$$
\begin{aligned}
&\bar{\sigma}_0\big|_{t_n} = {}^{\sigma}\alpha^{0}\big|_{t_n} - ({}^{\sigma}\alpha^{0}{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^{0}{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_2 = ({}^{\sigma}\alpha^{0}{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_3 = ({}^{\sigma}\alpha^{0}{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^1 = {}^{\sigma}\alpha^{1}\big|_{t_n} - ({}^{\sigma}\alpha^{1}{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^{1}{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_3^1 = ({}^{\sigma}\alpha^{1}{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_4^1 = ({}^{\sigma}\alpha^{1}{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^2 = {}^{\sigma}\alpha^{2}\big|_{t_n} - ({}^{\sigma}\alpha^{2}{}_{,2})_{t_n}(ii_{{}^{(1)}\gamma})_{t_n} - ({}^{\sigma}\alpha^{2}{}_{,3})_{t_n}(iii_{{}^{(1)}\gamma})_{t_n} \\
&{}^{\sigma}b_3^2 = ({}^{\sigma}\alpha^{2}{}_{,2})_{t_n} \quad ; \quad {}^{\sigma}b_4^2 = ({}^{\sigma}\alpha^{2}{}_{,3})_{t_n} \\
&{}^{\sigma}b_1^3 = \frac{\partial({}^{\sigma}\alpha^{0})}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^{\sigma}b_2^3 = \frac{\partial({}^{\sigma}\alpha^{1})}{\partial\bar{\theta}}\bigg|_{t_n} \quad ; \quad {}^{\sigma}b_3^3 = \frac{\partial({}^{\sigma}\alpha^{2})}{\partial\bar{\theta}}\bigg|_{t_n}
\end{aligned}
\tag{4.97}
$$

Using (4.97) we can write (4.96) as

$$
\begin{aligned}
[{}^{(0)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] &+ {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] \\
&+ {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_3^1(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] + {}^{\sigma}b_4^1(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma] \\
&+ {}^{\sigma}b_1^2[{}^{(1)}\gamma]^2 + {}^{\sigma}b_3^2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 + {}^{\sigma}b_4^2(iii_{{}^{(1)}\gamma})_{t_{n+1}}[{}^{(1)}\gamma]^2 \\
&+ {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] + {}^{\sigma}b_2^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma] + {}^{\sigma}b_3^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[{}^{(1)}\gamma]^2
\end{aligned}
\tag{4.98}
$$

in which the coefficients ${}^{\sigma}b_2$, ${}^{\sigma}b_3$, ${}^{\sigma}b_1^1$, ${}^{\sigma}b_3^1$, ${}^{\sigma}b_4^1$, ${}^{\sigma}b_1^2$, ${}^{\sigma}b_3^2$, ${}^{\sigma}b_4^2$ and ${}^{\sigma}b_1^3$, ${}^{\sigma}b_2^3$, ${}^{\sigma}b_3^3$ are functions of $\bar{\theta}_{t_n}$, $(ii_{{}^{(1)}\gamma})_{t_n}$ and $(iii_{{}^{(1)}\gamma})_{t_n}$ as evident from (4.97). These are the variable material coefficients that are dependent of the deformation during the evolution. $\bar{\sigma}_0\big|_{t_n}$ is the initial stress field associated with the configuration at time $t = t_n$. (4.98) are the constitutive equations for $[{}^{(0)}_{\ d}\bar{\sigma}]$ for *incompressible generalized Newtonian thermofluids with variable material coefficients*. This theory requires determination of a total of eleven variable material coefficients.

(a) Further assumptions and simplifications

The constitutive theory described by (4.98) can be further simplified if we make the following assumptions:

(i) We neglect the generator $[{}^{(1)}\gamma]^2$ all together in the development of the rate theory. Thus the terms in (4.98) containing the coefficients ${}^{\sigma}b_1^2$, ${}^{\sigma}b_3^2$, ${}^{\sigma}b_4^2$ and ${}^{\sigma}b_3^3$ can be deleted.

(ii) We also neglect all product terms in the current configuration at $t = t_{n+1}$ i.e., the products of generator $[{}^{(1)}\gamma]$ and its invariants can be removed from (4.98) including the products of $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ with $[{}^{(1)}\gamma]$.

(iii) When using simplifications (i) and (ii), we ensure that the material coefficients defined in (4.97) are not affected. With these assumptions, (4.98) reduces to

$$
[{}^{(0)}_{\ d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_2(ii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_3(iii_{{}^{(1)}\gamma})_{t_{n+1}}[I] + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.99}
$$

(4.99) is a much simplified constitutive theory for $[{}^{(0)}_{\;d}\bar{\sigma}]$. It only requires determination of four material coefficients.

(iv) If we further assume that the constitutive theory for $[{}^{(0)}_{\;d}\bar{\sigma}]$ to be linear in the components of $[{}^{(1)}\gamma]$, then $(ii_{{}^{(1)}\gamma})_{t_{n+1}}$ and $(iii_{{}^{(1)}\gamma})_{t_{n+1}}$ terms in (4.99) can be deleted.

$$[{}^{(0)}_{\;d}\bar{\sigma}] = \bar{\sigma}_0\big|_{t_n}[I] + {}^{\sigma}b_1^1[{}^{(1)}\gamma] + {}^{\sigma}b_1^3(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.100}$$

We redefine the material coefficients to conform to commonly used notations.

$$\begin{aligned} {}^{\sigma}b_1^1 &= 2\eta\big(\bar{\rho}_{t_n}\;,\;\bar{\theta}_{t_n}\;,\;(ii_{{}^{(1)}\gamma})_{t_n}\;,\;(iii_{{}^{(1)}\gamma})_{t_n}\big) = 2\eta_{t_n} \\ {}^{\sigma}b_1^3 &= -\alpha_{tm}\big(\bar{\rho}_{t_n}\;,\;\bar{\theta}_{t_n}\;,\;(ii_{{}^{(1)}\gamma})_{t_n}\;,\;(iii_{{}^{(1)}\gamma})_{t_n}\big) = -(\alpha_{tm})_{t_n} \end{aligned} \tag{4.101}$$

then we can write

$$[{}^{(0)}_{\;d}\bar{\sigma}] = \;\bar{\sigma}_0\big|_{t_n}[I] + 2\eta_{t_n}[{}^{(1)}\gamma] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.102}$$

(4.102) is the most simplified constitutive theory for $[{}^{(0)}_{\;d}\bar{\sigma}]$ for *incompressible generalized Newtonian thermofluids with variable material coefficients*. This theory requires determination of only two material coefficients. We note that (4.102) could be obtained using (4.42) by eliminating $\bar{\rho}$ and using the compressibility constraint tr$([{}^{(1)}\gamma]) = 0$.

***Remarks*:**

1. We note that (4.102) holds for the deformed configuration at time $t = t_{n+1}$ i.e., current configuration, thus $[{}^{(0)}_{\;d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ are defined at time $t = t_{n+1}$ but the coefficients η_{t_n} and $(\alpha_{tm})_{t_n}$ are obviously defined in the configuration at time $t = t_n$.

2. η is the *viscosity* of the fluid and α_{tm} is the *thermal modulus*.

3. By replacing $[{}^{(0)}_{\;d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ with the appropriate corresponding measures in the chosen

basis, $\left([{}_d\bar{\sigma}^{(0)}],\, [\gamma^{(1)}]\right)$, $\left([{}_d\bar{\sigma}_{(0)}],\, [\gamma_{(1)}]\right)$ and $\left([{}^{(0)}_{\ d}\bar{\sigma}^J],\, [{}^{(1)}\gamma^J]\right)$ we can obtain forms of (4.102) in contravariant basis, covariant basis and Jaumann, keeping in mind that the arguments of η and α_{tm} dependent on $[{}^{(1)}\gamma]$ are likewise modified. In (4.102), $ii_{{}^{(1)}\gamma}$ and $iii_{{}^{(1)}\gamma}$ need changes in η_{t_n} and $(\alpha_{tm})_{t_n}$ due to change of basis. As in case of compressible thermofluids, here also we note that

$$[{}^{(1)}\gamma] = [\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}] \tag{4.103}$$

Hence, the right side of (4.102) remains unaffected due to the change of basis, thus $[{}^{(0)}_{\ d}\bar{\sigma}] = [{}_d\bar{\sigma}^{(0)}] = [{}_d\bar{\sigma}_{(0)}] = [{}^{(0)}_{\ d}\bar{\sigma}^J] = [{}_d\bar{\sigma}]$ holds and we can write

$$[{}_d\bar{\sigma}] = \left.\bar{\sigma}_0\right|_{t_n}[I] + 2\eta_{t_n}[\bar{D}] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.104}$$

where $[{}_d\bar{\sigma}]$ is the deviatoric Cauchy stress tensor and the coefficients η_{t_n} and $(\alpha_{tm})_{t_n}$ become

$$\begin{aligned} \eta_{t_n} &= \eta\big(\bar{\theta}_{t_n}\,,\,(ii_{\bar{D}})_{t_n}\,,\,(iii_{\bar{D}})_{t_n}\big) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}\big(\bar{\theta}_{t_n}\,,\,(ii_{\bar{D}})_{t_n}\,,\,(iii_{\bar{D}})_{t_n}\big) \end{aligned} \tag{4.105}$$

(b) Variable transport properties or material coefficients

From (4.105), we note that η and α_{tm} can be functions of temperature during the evolution but their values must be determined based on $\bar{\theta}$ in the immediately preceding known configuration ($t = t_n$ in this case). (4.105) permit us to use experimentally and/or empirically determined relations for density and temperature dependent η, κ and α_{tm}. For example, in (4.104) and (4.105), power law and Sutherland models for $\eta_{t_n} = \eta_{t_n}(\bar{\theta}_{t_n})$ remain valid for the incompressible thermofluids as well.

Power law, Carreau-Yasuda models etc. for η

From (4.105), we note that viscosity η can be a function of the second and third principal invariants of $[\bar{D}]$ i.e., $ii_{\bar{D}}$ and $iii_{\bar{D}}$. This allows us to express η as a function of $ii_{\bar{D}}$ and $iii_{\bar{D}}$ using experimental and/or empirical relations between η and $ii_{\bar{D}}$ and $iii_{\bar{D}}$. Thus power law, Carreau-Yasuda models (described for compressible case) etc. for shear thinning and shear thickening fluids are justified and obviously have continuum mechanics basis.

***Remarks*:**

1. When η and α_{tm} in (4.104) show dependence on $\bar{\theta}_{t_n}$, $(ii_{\bar{D}})_{t_n}$ and $(iii_{\bar{D}})_{t_n}$, we refer to the fluid described by (4.104) as *incompressible generalized Newtonian thermoviscous fluid with variable transport properties*. In addition to power law and Carreau-Yasuda models for η_{t_n} as a function of $(ii_{\bar{D}})_{t_n}$, the other empirical and/or experimental relations are valid as well.

2. When η and α_{tm} only show dependence on $\bar{\theta}_{t_n}$ i.e., when

$$\begin{aligned} \eta_{t_n} &= \eta(\bar{\theta}_{t_n}) \\ (\alpha_{tm})_{t_n} &= \alpha_{tm}(\bar{\theta}_{t_n}) \end{aligned} \tag{4.106}$$

   Then, (4.104) and (4.106) describe *incompressible Newtonian thermoviscous fluids with variable transport properties*.

3. Constitutive relations (4.104) hold for the current configuration at time $t = t_{n+1}$. In (4.104) $[{}_d\bar{\sigma}]$ and $[\bar{D}]$ are in the current configuration at time $t = t_{n+1}$, however η_{t_n} and $(\alpha_{tm})_{t_n}$ are determined based on the known deformation field in the configuration corresponding to $t = t_n$. This is a consequence of the Taylor series expansion about the configuration at time $t = t_n$ of the coefficients ${}^{\sigma}\alpha^i$ used in the linear combination of the generators to define $[{}^{(0)}_{d}\bar{\sigma}]$. We remark that in the published works [22,38,39], this is not the case, but instead η and α_{tm} are treated as the functions of unknown deformation field in the current configuration at time $t = t_{n+1}$ i.e., instead of η_{t_n} and $(\alpha_{tm})_{t_n}$, these are replaced by

$$\begin{aligned} \eta_{t_{n+1}} &= \eta(\bar{\theta}_{t_{n+1}}\ ,\ (ii_{\bar{D}})_{t_{n+1}}\ ,\ (iii_{\bar{D}})_{t_{n+1}}) = \eta \\ (\alpha_{tm})_{t_{n+1}} &= \alpha_{tm}(\bar{\theta}_{t_{n+1}}\ ,\ (ii_{\bar{D}})_{t_{n+1}}\ ,\ (iii_{\bar{D}})_{t_{n+1}}) = \alpha_{tm} \end{aligned} \tag{4.107}$$

with the new definitions of η and α_{tm} in (4.107), (4.104) can be written as (no need to use the subscript t_{n+1} for the coefficients)

$$[{}_d\bar{\sigma}] = \left.\bar{\sigma}_0\right|_{t_n}[I] + 2\eta[\bar{D}] - \alpha_{tm}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I] \tag{4.108}$$

in which $[{}_d\bar{\sigma}]$, $[\bar{D}]$, η and α_{tm} are in the current configuration at time $t = t_{n+1}$. With (4.107), power law, Sutherland law for Newtonian case and power law, Carreau-Yasuda models for generalized Newtonian case change to (4.60) - (4.63). (4.108) and (4.60) - (4.63) are what is used currently in the published works. When the two configurations at times $t = t_n$ and $t = t_{n+1}$ are in close proximity of each other in terms of deformation field, using (4.108) and (4.61) - (4.63) may be justified but it is not supported by the derivation of the constitutive theory presented here.

4.9.2 Constitutive theory for the heat vector

Based on (4.92), it is obvious that the constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$ for the incompressible case remains the same as for the compressible case (section 4.7.2) except that $\bar{\rho}$ drops out in the entire derivation as it is not an argument tensor in (4.92). Details can be readily obtained from the derivation in section 4.7.2 and are not repeated here for the sake of brevity.

4.10 Numerical studies using non-linear constitutive equations for the deviatoric Cauchy stress tensor

If we decompose the Cauchy stress tensor in equilibrium stress and deviatoric Cauchy stress tensor, then the equilibrium stress is thermodynamic pressure (compressible case) or mechanical pressure (incompressible case), and the deviatoric Cauchy stress tensor becomes a dependent variable in the constitutive theory. In this section, we consider a subset of thermofluids of order $n = 1$. For these fluids with $n = 1$, the first convected time derivative of the strain tensor is an argument tensor of the dependent variables in the constitutive theory. The distinction between covariant and contravariant bases disappears in this case as the first convected time derivative of the stain tensor in the two bases is the same. We consider a constitutive theory for the deviatoric Cauchy stress tensor that is quadratic in $\boldsymbol{\gamma}^{(1)}$ or $\boldsymbol{\gamma}_{(1)}$ or $\bar{\boldsymbol{D}}$ and present comparison with the constitutive theory based on Newton's law of viscosity.

Since the description is understood to be Eulerian, we drop over bar $(\bar{\ })$ on all quantities for simplicity of notation. To conform to commonly used engineering notations we replace $\bar{x}_i$; $i = 1, 2, 3$ by x, y, z and $\bar{v}_i$; $i = 1, 2, 3$ by u, v, w and ${}^{(0)}_{\ d}\bar{\sigma}_{ij}$; $i, j = 1, 2, 3$ by τ_{ij} ; $i, j = 1, 2, 3$ in the mathematical model. We consider the following non-linear constitutive equation for the deviatoric Cauchy stress tensor $\boldsymbol{\tau}$:

$$[\tau] = \kappa \text{tr}([D])[I] + 2\eta[D] + \kappa_1 \text{tr}([D]^2)[I] + \eta_1[D]^2 \tag{4.109}$$

We consider fully developed flow of a constant viscosity imcompressible fluid between parallel plates as model problem. For this model problem the continuity equation is satisfied identically, hence the mathematical model describing the flow physics only consists of x- and y-momentum equations and the constitutive equation for the deviatoric Cauchy stress tensor if we assume isothermal flow.

$$\frac{\partial p}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} = 0 \tag{4.110}$$

$$\frac{\partial p}{\partial y} - \frac{\partial \tau_{yy}}{\partial y} = 0 \tag{4.111}$$

$$\tau_{xx} = \tau_{yy} = \big(\frac{\kappa_1}{2} + \frac{\eta_1}{4}\big)\left(\frac{\partial u}{\partial y}\right)^2 = \beta\left(\frac{\partial u}{\partial y}\right)^2 \tag{4.112}$$

$$\tau_{xy} = \eta \frac{\partial u}{\partial y} \tag{4.113}$$

where κ_1 and η_1 or $\beta = \frac{\kappa_1}{2} + \frac{\eta_1}{4}$ are the new coefficients associated with the non-linear terms with units of $\text{N.sec}^2/\text{m}^2$. Figure 4.1 shows a schematic and boundary conditions of the model problem using physical quantities. The plates are separated by a distance $2H$. The origin of the xy-coordinate is located at the center of the plates and the positive x-direction is the direction of the flow. The flow is pressure driven i.e. $\partial p/\partial x$ (negative) is specified.

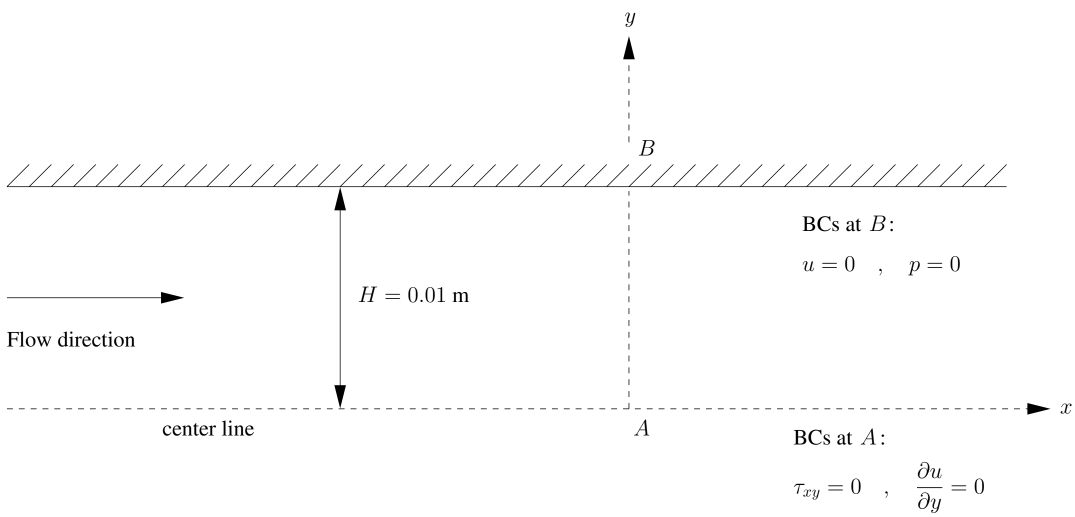


Figure 4.1: Schematic of 1-D fully developed flow between parallel plates (half domain)

Solution of the BVP:

In the numerical studies, we consider solutions of (4.110) - (4.113). The pressure gradient $\partial p/\partial x$ is specified and assumed constant, hence a theoretical solution is possible. Solving for τ_{xy} from (4.110) with the boundary condition $\tau_{xy} = 0$ at $y = 0$ gives

$$\tau_{xy} = \frac{\partial p}{\partial x} y \tag{4.114}$$

Substituting from (4.113) into (4.114), solving for u and using the boundary condition $u = 0$ at $y = H$ give

$$u = \frac{1}{2\eta} \frac{\partial p}{\partial x} (y^2 - H^2) \tag{4.115}$$

Substituting from (4.115) into (4.112) yields

$$\tau_{xx} = \tau_{yy} = \beta \left(\frac{1}{\eta} \frac{\partial p}{\partial x} \right)^2 y^2 \tag{4.116}$$

τ_{xx} and τ_{yy} are zero for the constitutive theory based on Newton's law of viscosity. In the numerical calculations of the solutions we consider air [60] at 15 °C with viscosity of

$$\eta = 0.0000179 \text{ N.sec/m}^2$$

The following pressure gradient values are chosen to present results

$$\frac{\partial p}{\partial x} = -0.1 \quad ; \quad -0.2 \quad ; \quad -0.3 \quad ; \quad -0.4 \quad ; \quad -0.5 \text{ Pa/m}$$

We consider the following values of the coefficient β

$$\beta = 0.0 \qquad ; \qquad 1.0 \times 10^{-5} \qquad ; \qquad 5.0 \times 10^{-5} \text{ N.sec}^2/\text{m}^2$$

When $\beta = 0.0$, (4.110) - (4.113) reduce to the mathematical model corresponding to fully developed flow between parallel plates for Newtonian fluids. The non-linear term does not affect the velocity profile or shear stress as expected based on (4.114) and (4.115), however with $\beta \neq 0$, the normal stresses $\tau_{xx} = \tau_{yy}$ are produced.

The material coefficient β in the non-linear constitutive equation for the deviatoric Cauchy stress tensor needs to be determined experimentally. The choice of the values of β is made so that with progressively increasing pressure gradient $\partial p/\partial x$ and as a consequence, increasing velocity gradient $\partial u/\partial y$, the constitutive equation based on Newton's law of viscosity remains valid up to some maximum value of $\partial p/\partial x$ i.e., for this range of $\partial p/\partial x$, the choices of numerical values of β have no significant influence on the stress field. Beyond certain value of $\partial p/\partial x$, the solutions obtained for non-zero β begin to differ from those obtained using the constitutive equation based on Newton's law of viscosity. We present numerical values in the following to demonstrate this.

Figures 4.2 - 4.4 show graphs of u, τ_{xy}, τ_{xx} and τ_{yy} versus distance y obtained using (4.114) - (4.116) with different values of pressure gradient $\partial p/\partial x$ and coefficient β. Figure 4.4 shows that for $\partial p/\partial x = -0.1$ and -0.2 Pa/m, the normal stresses corresponding to all three values of β are in good agreement confirming that within this range of $\partial p/\partial x$, the constitutive equation based on Newton's law of viscosity holds as non-zero β does not appreciably change τ_{xx} and τ_{yy} from their zero values corresponding to Newton's law of viscosity.

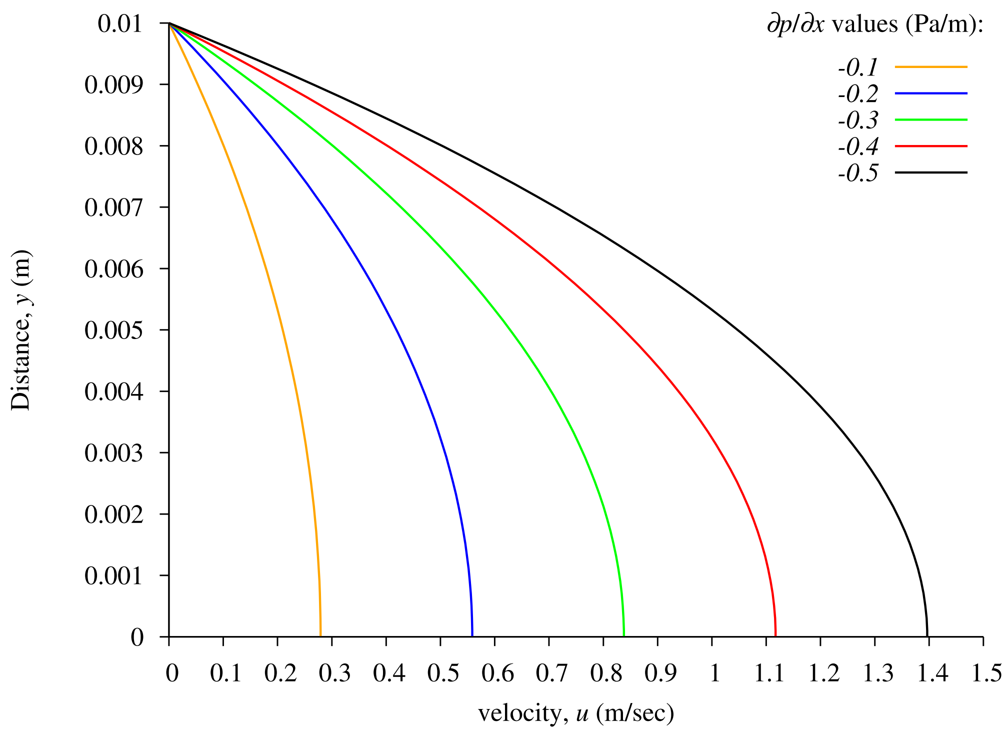


Figure 4.2: Velocity u versus distance y (for any value of β)

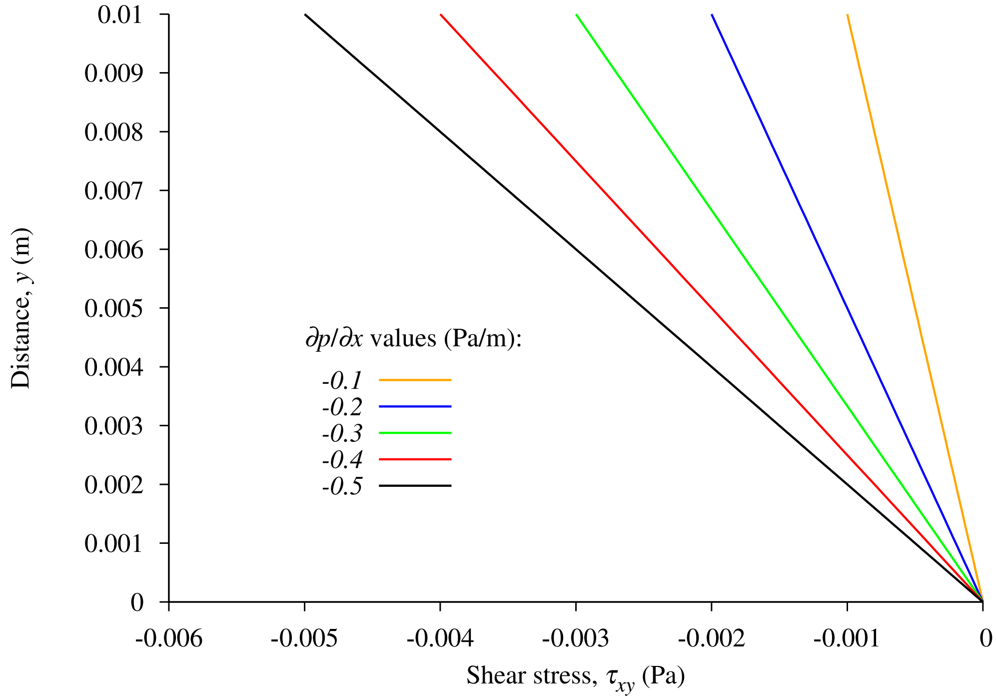


Figure 4.3: Shear stress versus distance y (for any value of β)

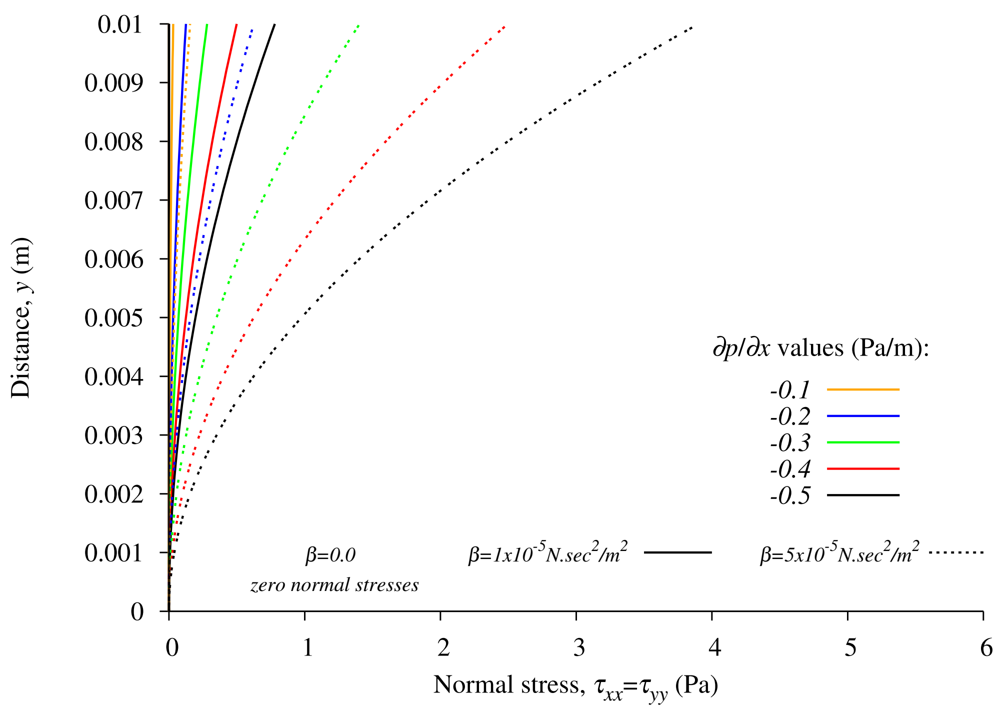


Figure 4.4: Normal stresses versus distance y

For $\partial p/\partial x = -0.3$ Pa/m, the normal stresses begin to differ from the predictions (zero normal stresses) corresponding to Newtonian fluids. As expected, the larger the value of β, the greater is the deviation of the normal stresses along the domain from the constitutive equation based on Newton's law of viscosity. For $\partial p/\partial x = -0.4$ Pa/m and -0.5 Pa/m, the normal stresses differ significantly compared to Newtonian fluids for which they are zero. Larger value of β results in larger normal stresses. An important point to note is that both constitutive equations predict zero normal stress differences. The study suggests that when flow rates and hence velocity gradients are high, the non-linear constitutive equation for the deviatoric Cauchy stress tensor may be a more realistic representation of the physics as opposed to the constitutive equation based on Newton's law of viscosity.

4.11 Summary

We have presented development of rate constitutive theories for incompressible as well as compressible ordered thermofluids in contravariant and covariant bases as well as using Jaumann rates. Based on the axiom of admissibility, all constitutive theories must satisfy conservation laws to ensure thermodynamic equilibrium of the deforming matter. Since conservation of mass, balance of momenta and energy equation only require existence of the stress field and heat vector, these are independent of the constitution of the matter. Thus the second law of thermodynamics (Clausius-Duhem inequality) must provide the basis for the constitutive theory.

The conditions resulting from the Clausius-Duhem inequality: (i) Show that η, specific entropy is deterministic from the Helmholtz free energy density and hence should not be considered as a dependent variable in the constitutive theory. Thus, the stress tensor, heat vector and the Helmholtz free energy density are the only dependent variables in the constitutive theory for the type of fluids considered here (ii) Provide a mechanism to determine the heat vector as a function of the temperature gradient vector and conductivity, i.e., Fourier heat condition law (iii) Do not provide a mechanism to determine constitutive equations for the total stress tensor. However, if the total Cauchy stress tensor is decomposed into equilibrium stress and deviatoric stress, then: (a) The equilibrium stress is deterministic from the entropy inequality and leads to thermodynamic pressure for compressible fluids and mechanical pressure in the case of incompressible fluids. The derivations are presented in this chapter. These hold regardless of the order of the thermofluid. (b) But the deviatoric Cauchy stress is not deterministic from the entropy inequality, however the entropy inequality does require the work expanded due to the deviatoric Cauchy stress to be positive. Thus the constitutive theory for ordered thermofluids reduces to deviatoric Cauchy stress tensor, heat vector and Helmholtz free energy density as dependent variables and their determination in terms of the argument tensors describing the flow physics in contravariant and covariant bases and using Jaumann rates.

Details of the contra- and co-variant bases, stress and strain measures, convected time derivatives of the stress and strain tensors in contra-, co-variant bases and using Jaumann rates, derivations of entropy inequality and the conditions resulting from it have been presented in references [55, 56]. It is shown that for compressible ordered thermofluids: (i) in contravariant basis, the argument tensors of deviatoric Cauchy stress $[{}_d\bar{\sigma}^{(0)}]$ and heat vector $\bar{\boldsymbol{q}}^{(0)}$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[\gamma^{(j)}]$; $j = 1, 2, \dots, n$, the convected time derivatives of orders $1, 2, \dots, n$ in the contravariant basis and for $\bar{\Phi}$, the argument tensors are $\bar{\rho}$ and $\bar{\theta}$. (ii) in covariant basis, the argument tensors of the deviatoric Cauchy stress $[{}_d\bar{\sigma}_{(0)}]$ and heat vector $\bar{\boldsymbol{q}}_{(0)}$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[\gamma_{(j)}]$; $j = 1, 2, \dots, n$, the convected time derivatives of orders $1, 2, \dots, n$ in the covariant basis and for $\bar{\Phi}$, the argument tensors are $\bar{\rho}$ and $\bar{\theta}$. (iii) Using Jaumann rates, the argument tensors of the deviatoric Jaumann stress tensor $[{}^{(0)}_{d}\bar{\sigma}^J]$ and heat vector ${}^{(0)}\bar{\boldsymbol{q}}^J$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$. For incompressible ordered thermofluids, density $\bar{\rho}$ in the current configuration is the same as in the reference configuration and hence it is no longer an argument of the dependent variables in the constitutive theory. Other arguments remain the same as for the compressible case.

The theory of generators and invariants is utilized to derive the general form of the constitutive theory for an n^{th} order 'ordered thermofluid' (both compressible and incompressible) in contravariant and covariant bases as well as using Jaumann rates. In this theory both the deviatoric Cauchy stress and the heat vectors are expressed as a linear combination of the combined generators of the argument tensors. The coefficients in this linear combination are functions of the combined invariants of the argument tensors in addition to $\bar{\rho}$ and $\bar{\theta}$ (in case of compressible fluids) or $\bar{\theta}$ (in case of incompressible fluids). The coefficients are determined by using their Taylor series expansion about the configuration at time $t = t_n$ assuming that $t = t_{n+1}$ corresponds to the current configuration) and retaining only up to linear terms in the combined invariants and the temperature. Explicit details are presented for second and first order 'ordered thermofluids'.

The general form of the ordered rate constitutive equations of order one is specialized and detailed derivations are presented for thermoviscous generalized Newtonian and Newtonian fluids (both compressible and incompressible). For such fluids, only the first convected time derivative of the strain tensor (Green or Almansi depending upon co- or contra-variant basis or Jaumann rate) remains as argument tensor for the deviatoric stress (in addition to density and temperature). The heat vector does not contain the first convected time derivative of the strain tensor as an argument. Based on the theories presented in this chapter for ordered thermofluids we make the following remarks.

1. For ordered thermofluids of order greater than or equal to two (i.e., when $[\gamma^{(2)}]$, $[\gamma^{(3)}]$, $\dots$ or $[\gamma_{(2)}]$, $[\gamma_{(3)}]$, $\dots$ or $[{}^{(2)}\gamma^J]$, $[{}^{(3)}\gamma^J]$, $\dots$ are argument tensors), the contra- and covariant stress measures as well as Jaumann stress tensor are not the same, i.e., in this case $[{}_d\bar{\sigma}^{(0)}] \neq [{}_d\bar{\sigma}_{(0)}] \neq [{}^{(0)}_{d}\bar{\sigma}^J]$ even though they all are in x-frame with the same dyads. The same is true for the constitutive theory for the heat vector when the convected derivatives of the strain tensor of orders higher than one are the argument tensors i.e., for this case $\bar{\boldsymbol{q}}^{(0)} \neq \bar{\boldsymbol{q}}_{(0)} \neq {}^{(0)}\bar{\boldsymbol{q}}^J$.

2. For ordered thermofluids of order one, i.e., when $[\gamma^{(1)}]$ or $[\gamma_{(1)}]$ or $[{}^{(1)}\gamma^J]$ are argument tensors, the contra- and co-variant stress measures are the same, i.e., in this case $[{}_d\bar{\sigma}^{(0)}] = [{}_d\bar{\sigma}_{(0)}] = [{}^{(0)}_{\ d}\bar{\sigma}^J]$. For such fluids, the distinction between contra- and co-variant bases and Jaumann rates disappears and we may simply say deviatoric Cauchy stress $[{}_d\bar{\sigma}]$ as opposed to contra- and co-variant deviatoric Cauchy stress tensor or Jaumann stress tensor in the x-frame. Thus, for generalized Newtonian and Newtonian fluids (both compressible and incompressible) the covariant and contravariant stress measures and Jaumann stress measure are the same.

3. Based on the theory of generators and invariants, the constitutive theory for heat vector for an ordered thermofluid is much more complex (even for thermofluids of order one due to the dependence of the heat vector on the combined generators of $[\gamma^{(1)}]$, $\bar{\boldsymbol{g}}$ or $[\gamma_{(1)}]$, $\bar{\boldsymbol{g}}$ or $[{}^{(1)}\gamma^J]$, $\bar{\boldsymbol{g}}$) compared to Fourier heat conduction law which requires that the heat vector not be dependent on $[\gamma^{(1)}]$ or $[\gamma_{(1)}]$ or $[{}^{(1)}\gamma^J]$. The constitutive theory for the heat vector based on the combined generators of $[\gamma^{(1)}]$, $\bar{\boldsymbol{g}}$ or $[\gamma_{(1)}]$, $\bar{\boldsymbol{g}}$ or $[{}^{(1)}\gamma^J]$, $\bar{\boldsymbol{g}}$ is perhaps more realistic for fluids as it accounts for velocity gradients. However, their use will require experimental determination of additional material coefficients.

4. It is shown that the constitutive theory for the deviatoric Cauchy stress for generalized Newtonian (compressible as well as incompressible) thermofluids has continuum mechanics basis and is derivable using rate constitutive theory of order one with further assumptions and simplifications. The derivations support variable transport properties (i.e., functions of temperature and density) a well as dependence of the transport properties on the invariants of $\bar{\boldsymbol{D}}$ that form the basis for power law and Carreau-Yasuda models (and others) for shear rate dependent viscosity. Thus power law and Carreau-Yasuda models are not "useful empiricism" [22] but have continuum mechanics foundation as shown here.

5. A significant point to note in this chapter is that determination of coefficients used in the linear combination of the generators to express the deviatoric stress tensor or heat vector

requires use of Taylor series about the configuration at $t = t_n$ when $t = t_{n+1}$ is the current configuration. This automatically forces the determination of the coefficients in the configuration at time $t = t_n$ and not at $t = t_{n+1}$ corresponding to the current configuration. In all presently used works, this is not the case. Variable transport properties as well as material coefficients dependent on the invariants of $\bar{\boldsymbol{D}}$ are all expressed using the current configuration. This may be justified when the configurations at $t = t_n$ and $t = t_{n+1}$ are in close proximity in terms of deformation field but can not be supported by the derivations presented in this work.

6. All constitutive theories for ordered thermofluids in contra- or co-variant bases and using Jaumann rates are in fact rate constitutive theories. In the case of the constitutive theory in the contravariant basis, we express the convected time derivative of order zero of the contravariant deviatoric Cauchy stress tensor $[{}_d\bar{\sigma}^{(0)}]$ in terms of the convected time derivatives of various orders of the Almansi strain tensor in the contravariant basis. Likewise, for the constitutive theory in the covariant basis, we express the convected time derivative of order zero of the covariant deviatoric Cauchy stress tensor $[{}_d\bar{\sigma}_{(0)}]$ in terms of the convected time derivatives of various orders of the Green's strain tensor in the covariant basis. In the case of Jaumann stress tensor $[{}^{(0)}_{d}\bar{\sigma}^J]$ we use Jaumann rates.

   Thus, the constitutive equations for generalized Newtonian and Newtonian thermofluids are indeed rate constitutive equations. The distinction between co- and contra-variant measures and the measures based on Jaumann rates disappears for such fluids due to the fact that the convected time derivative of order one of the Green's strain tensor is the same as the convected time derivative of order one of the Almansi strain tensor as well as the Jaumann strain rate.

7. It is significant to note that based on Surana et al. [30], when the deformation is finite, only the constitutive theories derived using contravariant basis remain valid. As the magnitude of the deformation increases, the constitutive theories in covariant basis and others become

progressively more spurious as these use stress measures that do not correspond to the true deformed tetrahedron in the current configuration.

8. The condition of positive work expanded resulting from the entropy inequality must be satisfied by all rate constitutive equations presented here.

9. Numerical studies are presented for a subset of thermofluids of order $n = 1$. We consider a constitutive theory for the deviatoric Cauchy stress tensor that is quadratic in $\bar{\boldsymbol{D}}$ and present comparison with the constitutive theory based on Newton's law of viscosity. We consider fully developed flow of a constant viscosity imcompressible fluid (air) between parallel plates as model problem. The new coefficient β associated with the non-linear terms must be determined experimentally. The pressure gradient $\partial p/\partial x$ is specified and assumed constant, hence a theoretical solution is possible.

10. The choice of the values of β is made so that with progressively increasing pressure gradient $\partial p/\partial x$ and as a consequence, increasing velocity gradient $\partial u/\partial y$, the constitutive equation based on Newton's law of viscosity remains valid up to some maximum value of $\partial p/\partial x$ i.e., for this range of $\partial p/\partial x$, the choices of numerical values of β have no significant influence on the normal stress field. Beyond certain value of $\partial p/\partial x$, the solutions obtained for non-zero β begin to differ from those obtained using the constitutive equation based on Newton's law of viscosity. The study suggests that when flow rates and hence velocity gradients are high, the non-linear constitutive equation for the deviatoric Cauchy stress tensor may be a more realistic representation of the physics as opposed to the constitutive equation based on Newton's law of viscosity.

11. Since the constitutive theories in this chapter are based on combined generators and invariants of the argument tensors of the dependent variable, strictly speaking these might be viewed to lack thermodynamic basis (as these are not derived using entropy inequality). However, the theories do have continuum mechanics foundation as these satisfy the axioms of the constitutive theory.

The material presented in this chapter provides a completely general and unified theory for ordered thermofluids from which specialized fluid behaviors such as generalized Newtonian fluids, Newtonian fluids etc. can be easily derived as shown in this chapter. It is demonstrated that the distinction between contra- and co-variant bases and Jaumann rates is critical for ordered thermofluids of order greater than or equal to two.

Chapter 5

Rate Constitutive Theories in Eulerian Description for Ordered Thermoviscoelastic Fluids - Polymers

5.1 Introduction

In this chapter we consider development of rate constitutive theories for compressible as well as in incompressible homogeneous and isotropic ordered thermoviscoelastic fluids, i.e., polymeric fluids, in Eulerian description. The polymeric fluids are considered as ordered thermoviscoelastic fluids in which the stress rate of a desired order, i.e., the convected time derivative of a desired order 'm' of the chosen deviatoric Cauchy stress tensor, and the heat vector are functions of density, temperature, temperature gradient, convected time derivatives of the chosen strain tensor up to any desired order 'n' and the convected time derivative of up to orders '$m-1$' of the chosen deviatoric Cauchy stress tensor. The polymeric fluids described by these constitutive theories will be referred to as *ordered thermoviscoelastic fluids* due to the fact that the constitutive theories are dependent on the orders 'm' and 'n' of the convected time derivatives of the deviatoric Cauchy stress and conjugate strain tensors. The highest orders of the convected time derivative of the deviatoric Cauchy stress and strain tensors define the orders of the polymeric fluid.

We consider general ordered rate constitutive theories for thermoviscoelastic fluids in co- and contra-variant bases as well as using Jaumann rates based on the principles and axioms of continuum mechanics. The Maxwell, Giesekus and Oldroyd-B constitutive models used currently are derived as special cases of the general ordered rate theories. The general derivation of the rate constitutive theories for the deviatoric Cauchy stress and the heat vector are specialized to derive upper convected (contravariant basis), lower convected (covariant basis) and Jaumann rate constitutive equations commonly used for Maxwell, Giesekus and Oldroyd-B fluids.

5.2 Rate Constitutive Theories in Eulerian Description

In chapter 2 we had considered entropy inequality in Lagrangian description to conclude that Φ, $\boldsymbol{\sigma}^*$, $\boldsymbol{q}$ and η must be the dependent variables in the constitutive theories. We considered $[J]$, $[\dot{J}]$, θ and $\boldsymbol{g}$ as arguments of the dependent variables in the constitutive theories. Using entropy inequality in Lagrangian description it was concluded that: (i) Φ is not a function of $[\dot{J}]$ (ii) Φ is not a function of $\boldsymbol{g}$ either (iii) η is not a dependent variable in the constitutive theory (iv) consideration of $\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - \sigma^*_{ki} = 0$ and $\underset{\sim}{q_i} \underset{\sim}{g_i} \leq 0$ is inappropriate due to the fact that in this case $\boldsymbol{\sigma}^*$ is not a function of $[\dot{J}]$ as Φ is not a function of $[\dot{J}]$, which is contrary to the assumption that $\boldsymbol{\sigma}^*$ depends on $[\dot{J}]$. Thus, entropy inequality does not provide any further means of determining the constitutive theories for neither $\boldsymbol{\sigma}^*$ nor $\boldsymbol{q}$. It was shown that by considering stress decomposition into equilibrium and deviatoric stress i.e. $\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^* + {}_d\boldsymbol{\sigma}^*$ in which ${}_e\boldsymbol{\sigma}^*$ is not a function of $[\dot{J}]$ and ${}_d\boldsymbol{\sigma}^*$ becomes zero when $[\dot{J}]$ and $\boldsymbol{g}$ are zero, and using the conditions resulting from the entropy inequality, that ${}_d\sigma^*_{ki} \dot{J}_{ik} > 0$ and $\underset{\sim}{q_i} \underset{\sim}{g_i} \leq 0$ must hold, which gave us $\Phi = \Phi([J], \theta)$, $\sigma^*_{ij} = \rho_0 \frac{\partial \Phi}{\partial J_{ji}} + {}_d\sigma^*_{ij}([J], [\dot{J}], \theta, \boldsymbol{g})$ and $\boldsymbol{q} = \boldsymbol{q}([J], [\dot{J}], \theta, \boldsymbol{g})$. Due to frame invariance considerations, dependence on $[J]$ must be replaced by I_J, $I\!I_J$, $I\!I\!I_J$ and $[\dot{J}]$ can be replaced by I_J, $I\!I_J$, $I\!I\!I_J$, $[D]$. These arguments hold in Lagrangian description. However, in Eulerian description, material point displacements are not known, hence $[J]$ is not deterministic but $\mathit{III}_J = \det[J] = \rho_0/\bar{\rho}$ i.e. dependence on III_J can be replaced by $\bar{\rho}$, but dependence on I_J and II_J can not be considered. Thus, in Eulerian description we consider the following in contravariant basis

$$\begin{aligned}
&\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \\
&[\bar{\sigma}^{(0)}] = [\,{}_e\bar{\sigma}^{(0)}\,] + [\,{}_d\bar{\sigma}^{(0)}(\bar{\rho}, [\bar{D}], \bar{\theta}, \bar{\boldsymbol{g}})\,] \quad ; \quad [{}_e\sigma^*]^T = \rho_0 \frac{\partial \Phi}{\partial [J]} \\
&\bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}^{(0)}(\bar{\rho}, [\bar{D}], \bar{\theta}, \bar{\boldsymbol{g}})
\end{aligned} \tag{5.1}$$

For compressible matter, equilibrium stress is a function $\bar{\Phi}$ and thus it is deterministic from the deformation field. For incompressible matter, equilibrium stress is also derived from the entropy inequality in conjunction with incompressibility constraint, however, equilibrium stress is not a function of $\bar{\Phi}$ and thus it is not deterministic from the deformation field. It was shown that in both cases, equilibrium stress is independent of the basis. We make the following remarks:

(1) The second law of thermodynamics only restricts the work expanded due to the deviatoric stress to be positive but provides no mechanism for determining the constitutive theory for the deviatoric stress. In addition, $\underset{\sim}{q_i}\underset{\sim}{g_i} \leq 0$ must also hold.

(2) The theory of generators and invariants [3–21] provides a continuum mechanics foundation to derive constitutive equations for the deviatoric Cauchy stress tensor and heat vector in which we determine combined generators of the argument tensors that form *integrity or minimal basis*. The dependent variables in the constitutive theories are expressed as linear combinations of the combined generators of the argument tensors. The coefficients used in the linear combinations are functions of $\bar{\rho}$, $\bar{\theta}$, and the combined invariants of the argument tensors in the current configuration which, using the *axiom of smooth neighborhood*, are determined by using their Taylor series expansion about a previously known configuration.

5.3 Thermoviscoelastic fluids: dependent variables in the constitutive theories and their argument tensors

Let $[\gamma^{(j)}]$, $[\gamma_{(j)}]$, $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$ be the convected time derivatives of order $1, 2, \dots, n$ of the Almansi strain tensor $[\bar{\varepsilon}]$, Green's strain tensor $[\varepsilon]$ and Jaumann strain rates. These are *fundamental kinematic symmetric tensors of rank two*. Likewise, let $[{}_d\bar{\sigma}^{(k)}]$, $[{}_d\bar{\sigma}_{(k)}]$ and $[{}^{(k)}_{\ d}\bar{\sigma}^J]$; $k = 0, 1, \dots, m$ be convected time derivatives of orders $0, 1, \dots, m$ of the Cauchy stress tensors in the three bases. These are *fundamental symmetric tensors of rank two*. We note that $[\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}]$. Using the new notation introduced in chapter 2 we can generalize (5.1) by replacing $[\bar{D}]$ with $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ depending upon whether the basis is contra- or co-variant or Jaumann basis. Consider current configuration at time $t = t_{n+1}$ and let $[{}^{(k)}_{\ d}\bar{\sigma}\ ]$; $k = 0, 1, \dots, m$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ be the convected time derivatives of the deviatoric Cauchy stress tensor, heat vector and the corresponding convected time derivatives of the strain tensor in the chosen basis. Let $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$ be the density, temperature and temperature gradient in the current configuration.

From the Maxwell model, Giesekus model, Oldroyd-B model etc. we note that these models contain convected time derivatives of orders one and zero of the stress tensor. *Thus these must be derivable by considering the first convected time derivative of the deviatoric Cauchy stress tensor as a dependent variable in the constitutive theory in which the convected time derivative of order zero of the deviatoric Cauchy stress tensor is an argument tensor* (see later sections). In the constitutive theories presented in this chapter for thermoviscoelastic fluids, we generalize this concept and consider the convected time derivative of order 'm' of the chosen deviatoric Cauchy stress tensor (co- or contra-variant basis or Jaumann) as a dependent variable in the constitutive theories with convected time derivatives of up to order '$m - 1$' of the same deviatoric stress tensor as its arguments in addition to the other argument tensors.

Thus, in the rate constitutive theories presented in this chapter for thermoviscoelastic fluids, we consider $[{}^{(m)}\bar{\sigma}]$, ${}^{(0)}\bar{\boldsymbol{q}}$, $\bar{\Phi}$ as dependent variables in the constitutive theories. $[{}^{(k)}_{\ d}\bar{\sigma}\ ]$; $k = 0, 1, \dots, m-1$, $[{}^{(j)}\gamma\ ]$; $j = 1, 2, \dots, n$, $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$ are argument tensors of $[{}^{(m)}_{\ d}\bar{\sigma}\ ]$ and ${}^{(0)}\bar{\boldsymbol{q}}$. The argument tensors of $\bar{\Phi}$ for compressible case are $\bar{\rho}$ and $\bar{\theta}$ and for incompressible case, only $\bar{\theta}$ is the argument tensor of $\bar{\Phi}$. Hence, we have the following for the compressible and incompressible thermoviscoelastic fluids considered here (choice III, chapter 2).

Compressible thermoviscoelastic fluids:

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\rho}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\big) \tag{5.2}$$

$$[{}^{(0)}\bar{\sigma}\ ] = \big[{}^{(0)}_{\ e}\bar{\sigma}\big(\bar{\rho}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\big)\big] + \big[{}^{(0)}_{\ d}\bar{\sigma}\ \big] \tag{5.3}$$

$$\begin{aligned} [{}^{(m)}_{\ d}\bar{\sigma}\ ] = \big[{}^{(m)}_{\ d}\bar{\sigma}\big(\bar{\rho}(\bar{\boldsymbol{x}}, t)\ ,\ [{}^{(k)}_{\ d}\bar{\sigma}(\bar{\boldsymbol{x}}, t)]\ ;\ k = 0, 1, \dots, m-1\ ,& \\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)]\ ;\ j = 1, 2, \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t)\big)\big]& \end{aligned} \tag{5.4}$$

$$\begin{aligned} {}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\ \bar{\rho}(\bar{\boldsymbol{x}}, t)\ ,\ [{}^{(k)}_{\ d}\bar{\sigma}(\bar{\boldsymbol{x}}, t)]\ ;\ k = 0, 1 \dots, m-1\ ,& \\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}}, t)]\ ;\ j = 1, 2 \dots, n\ ,\ \bar{\theta}(\bar{\boldsymbol{x}}, t)\ ,\ \bar{\boldsymbol{g}}(\bar{\boldsymbol{x}}, t)\big)& \end{aligned} \tag{5.5}$$

in which equilibrium stress $[{}^{(0)}_{\ e}\bar{\sigma}]$ is thermodynamic pressure $\bar{p}(\bar{\rho}, \bar{\theta})[I]$ and it is independent of the basis. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\rho}, \bar{\theta})$ can be replaced by $-\bar{p}(\bar{\rho}, \bar{\theta})$ (see chapter 2, section 2.6 for derivation).

Incompressible thermoviscoelastic fluids:

$$\bar{\Phi} = \bar{\Phi}\big(\bar{\theta}(\bar{\boldsymbol{x}},t)\big) \tag{5.6}$$

$$\big[{}^{(0)}\bar{\sigma}\,\big] = \big[{}^{(0)}_{\;e}\bar{\sigma}\big(\bar{\theta}(\bar{\boldsymbol{x}},t)\big)\big] + \big[{}^{(0)}_{\;d}\bar{\sigma}\,\big] \tag{5.7}$$

$$\begin{aligned}\big[{}^{(m)}_{\;d}\bar{\sigma}\,\big] = \big[{}^{(m)}_{\;d}\bar{\sigma}\big(\,[{}^{(k)}_{\;d}\bar{\sigma}(\bar{\boldsymbol{x}},t)]\,;\,k=0,1,\dots,m-1\,,\\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\,;\,j=1,2,\dots,n\,,\;\bar{\theta}(\bar{\boldsymbol{x}},t)\,,\;\bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big)\big]\end{aligned} \tag{5.8}$$

$$\begin{aligned}{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\,[{}^{(k)}_{\;d}\bar{\sigma}(\bar{\boldsymbol{x}},t)]\,;\,k=0,1,\dots,m-1\,,\\ [{}^{(j)}\gamma(\bar{\boldsymbol{x}},t)]\,;\,j=1,2\dots,n\,,\;\bar{\theta}(\bar{\boldsymbol{x}},t)\,,\;\bar{\boldsymbol{g}}(\bar{\boldsymbol{x}},t)\big)\end{aligned} \tag{5.9}$$

where equilibrium stress $[{}^{(0)}_{\;e}\bar{\sigma}]$ is mechanical pressure $\bar{p}(\bar{\theta})[I]$ and it also is independent of the basis. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\theta})$ can be replaced by $-\bar{p}(\bar{\theta})$ (see chapter 2, section 2.6 for derivation).

The constitutive theories for $[{}^{(m)}_{\;d}\bar{\sigma}\,]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ are derived using (5.4) and (5.5) or (5.8) and (5.9), and can be converted to contravariant basis, covariant basis or Jaumann rate by replacing $[{}^{(m)}_{\;d}\bar{\sigma}\,]$, $[{}^{(0)}_{\;d}\bar{\sigma}\,]$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]\,;\,j=1,2,\dots,n$ with the appropriate measures in the chosen basis. In the following sections we consider details of development of rate constitutive theories for deviatoric Cauchy stress and heat vector for both compressible and incompressible thermoviscoelastic fluids.

5.4 Rate constitutive theory of orders '*m*' and '*n*' for the deviatoric Cauchy stress tensor and the heat vector: compressible thermoviscoelastic fluids

Consider a deforming volume of compressible thermoviscoelastic fluid at time $t = t_{n+1}$, the current configuration. We derive the rate constitutive theory of orders 'm' and 'n' for the deviatoric

Cauchy stress tensor $[{}^{(0)}_{\;d}\bar{\sigma}]$ and heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ using ((5.4) and (5.5))

$$
\begin{aligned}
&[{}^{(m)}_{\;d}\bar{\sigma}] = [{}^{(m)}_{\;d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(k)}_{\;d}\bar{\sigma}]\ ;\ k=0,1,\dots,m-1\ ,\ [{}^{(j)}\gamma]\ ;\ j=1,2,\dots,n\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \\
&{}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}\big(\bar{\rho}\ ,\ [{}^{(k)}_{\;d}\bar{\sigma}]\ ;\ k=0,1,\dots,m-1\ ,\ [{}^{(j)}\gamma]\ ;\ j=1,2,\dots,n\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}\big)
\end{aligned}
\tag{5.10}
$$

5.4.1 Constitutive theory of orders '*m*' and '*n*' for $[{}^{(m)}_{\;d}\bar{\sigma}]$

Let $[{}^{\sigma}\underset{\sim}{G}^i]\ ;\ i=1,2,\dots,N$ be the combined generators (of $[{}^{(m)}_{\;d}\bar{\sigma}]$) of the argument tensors $[{}^{(k)}_{\;d}\bar{\sigma}]\ ; k=0,1,\dots,m-1$, $[{}^{(j)}\gamma]\ ; j=1,2,\dots,n$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two, and let ${}^{q\sigma}\underset{\sim}{I}^j\ ; j=1,2,\dots,M$ be the combined invariants of the same argument tensors. Then, we can express $[{}^{(m)}_{\;d}\bar{\sigma}]$ as a linear combination of the generators $[{}^{\sigma}\underset{\sim}{G}^i]\ ; i=1,2,\dots,N$ and the identity tensor $[I]$ in the current configuration at time $t=t_{n+1}$.

$$
[{}^{(m)}_{\;d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + \sum_{i=1}^{N}{}^{\sigma}\alpha^i[{}^{\sigma}\underset{\sim}{G}^i]
\tag{5.11}
$$

The coefficients ${}^{\sigma}\alpha^i\ ;\ i=0,1,\dots,N$ in (5.11) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^j\ ; j=1,2,\dots,M$ in the current configuration at time $t=t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}\ ; j=1,2,\dots,M$. To determine the coefficients ${}^{\sigma}\alpha^i\ ; i=0,1,\dots,N$ in (5.11) related to the configuration at time $t=t_{n+1}$, we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i=0,1,\dots,N$ about the configuration at time $t=t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j\ ;\ j=1,2,\dots,M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{M}\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n}\big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\ ;\ i=0,1,\dots,N
\tag{5.12}
$$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}\ ;\ j=1,2,\dots,M$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}\ ; i=0,1,\dots,N$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\ ; j=1,2,\dots,M$ whereas in (5.12), ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\ ,\ \bar{\theta}_{t_{n+1}}\ ,$ $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j=1,2,\dots,M\big)\ ; i=0,1,\dots,N$. When (5.12) is substituted in (5.11), we obtain the final form of the most general rate constitutive theory of orders (m,n) for $[{}^{(m)}_{\;d}\bar{\sigma}]$ for compressible thermoviscoelastic fluids. This theory uses integrity and hence is complete.

5.4.2 Constitutive theory of orders '*m*' and '*n*' for ${}^{(0)}\bar{\boldsymbol{q}}$

Let $\{{}^q\underset{\sim}{G}^i\}$; $i = 1, 2, \dots, \widetilde{N}$ be the combined generators (of ${}^{(0)}\bar{\boldsymbol{q}}$) of the argument tensors $[{}^{(k)}_{\;\,d}\bar{\sigma}]$; $k = 0, 1, \dots, m-1$, $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one. The combined invariants of these argument tensors obviously remain the same as for $[{}^{(m)}_{\;\,d}\bar{\sigma}]$ i.e., ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$. Then we can express ${}^{(0)}\bar{\boldsymbol{q}}$ as a linear combination of $\{{}^q\underset{\sim}{G}^i\}$; $i = 1, 2, \dots, \widetilde{N}$ in the current configuration at time $t = t_{n+1}$.

$$
{}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{\widetilde{N}} {}^q\alpha^i \{{}^q\underset{\sim}{G}^i\} \tag{5.13}
$$

The absence of unit vector in (5.13) *as a generator is due to the fact that uniform temperature field does not contribute to* ${}^{(0)}\bar{\boldsymbol{q}}$. *The negative sign in* (5.13) *is because a positive* ${}^{(0)}\bar{\boldsymbol{q}}$ *in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter*. The coefficients ${}^q\alpha^i$; $i = 1, 2, \dots, \widetilde{N}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j; j = 1, 2, \dots, M$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \dots, M$. To determine the coefficients ${}^q\alpha^i$; $i = 1, 2, \dots, \widetilde{N}$ (in the current configuration at time $t = t_{n+1}$) in (5.13), we consider Taylor series expansion of each ${}^q\alpha^i$; $i = 1, 2, \dots, \widetilde{N}$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$
{}^q\alpha^i = {}^q\alpha^i\big|_{t_n} + \sum_{j=1}^{M} \frac{\partial({}^q\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \Big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n} \Big) + \frac{\partial({}^q\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 1, 2, \dots, \widetilde{N} \tag{5.14}
$$

${}^q\alpha^i\big|_{t_n}$, $\frac{\partial({}^q\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, M$ and $\frac{\partial({}^q\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1, 2, \dots, \widetilde{N}$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, M$ whereas ${}^q\alpha^i = {}^q\alpha^i\big(\bar{\rho}_{t_n}\,,\ \bar{\theta}_{t_n}\,,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, M\,,\ \bar{\theta}_{t_{n+1}}\,,\ ({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \dots, M\big)$; $i = 1, 2, \dots, \widetilde{N}$ in (5.14). When (5.14) is substituted in (5.13), we obtain the final expression for the most general rate constitutive theory of orders (m, n) for ${}^{(0)}\bar{\boldsymbol{q}}$ for compressible thermoviscoelastic fluids. This theory uses integrity and hence is complete.

5.4.3 Remarks:

1. In sections 5.4.1 - 5.4.2 we have presented rate constitutive theories of orders (m,n) for the deviatoric Cauchy stress tensor and heat vector using $[{}^{(m)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ as dependent variables with $[{}^{(k)}_{d}\bar{\sigma}]$; $k = 0,1,\dots,m-1$ stress rate tensors and $[{}^{(j)}\gamma]$; $j = 1,2,\dots,n$ strain rate tensors as their argument tensors, in addition to $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$. Hence, these developments are independent of the basis.

2. By replacing $[{}^{(k)}_{d}\bar{\sigma}]$; $k = 0,1,\dots,m$, ${}^{(0)}\bar{\boldsymbol{q}}$ and $[{}^{(j)}\gamma]$; $j = 1,2,\dots,n$ with the appropriate corresponding measures in the chosen basis, we can readily obtain the rate theories of orders (m,n) for the deviatoric Cauchy stress tensor and the heat vector in the basis of choice. More specifically we use the following measures:

$$
\begin{array}{llll}
\text{Contravariant:} & [{}_{d}\bar{\sigma}^{(k)}] \; ; \; k = 0,1,\dots,m & , \; \bar{\boldsymbol{q}}^{(0)} \; , [\gamma^{(j)}] & ; \; j = 1,2,\dots,n \\
\text{Covariant:} & [{}_{d}\bar{\sigma}_{(k)}] \; ; \; k = 0,1,\dots,m & , \; \bar{\boldsymbol{q}}_{(0)} \; , [\gamma_{(j)}] & ; \; j = 1,2,\dots,n \\
\text{Jaumann:} & [{}^{(k)}_{d}\bar{\sigma}^{J}] ; \; k = 0,1,\dots,m & , \; {}^{(0)}\bar{\boldsymbol{q}}^{J}, [{}^{(j)}\gamma^{J}] ; & \; j = 1,2,\dots,n
\end{array}
\tag{5.15}
$$

3. Since the tensor $\bar{\boldsymbol{g}}$ is independent of the choice of basis, the combined generators and the combined invariants used in sections 5.4.1 - 5.4.2 only need to be redefined using the convected rates $[\gamma^{(j)}]$; $j = 1,2,\dots,n$, $[\gamma_{(j)}]$; $j = 1,2,\dots,n$, $[{}^{(j)}\gamma^{J}]$; $j = 1,2,\dots,n$ and the corresponding convected rates of the Cauchy stress tensor and Jaumann stress tensor to obtain the contravariant, covariant and Jaumann rate theories of orders 'm' and 'n' for the deviatoric Cauchy stress tensor and heat vector.

4. In the final expression for $[{}^{(m)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ containing sum of many group of terms, we consider the following arrangement (in general).

   (a) In each group, *the terms that are defined in the configuration at time $t = t_n$ are grouped to define material coefficients*.

   (b) With choice (a), the expression for $[{}^{(m)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ will now consist of the sum of the material coefficients defined in the configuration at $t = t_n$ multiplied with the generators and/or invariants in the current configuration at time $t = t_{n+1}$ for which the deformation is not known.

(c) The material coefficients defined in (a) will be functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$.

We follow this arrangement (as far as possible) in all subsequent derivations. These theories use integrity and hence are complete but are too complicated and impractical as they contain too many material coefficients that must be determined experimentally and/or empirically. Details of (a) - (c) are clearly shown in sections 5.5 and 5.6.

5. Dependence of the coefficients in the final form of the constitutive equations for the deviatoric Cauchy stress tensor and heat vector on $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$ permits variable material coefficients during the evolution. Thus material coefficients can be functions of density and temperature during the evolution for which experimental and/or empirical relations such as power law, Sutherland law etc. are justified. Furthermore, dependence of the coefficients on the invariants $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$ permits complex description of material coefficients on the deformation field. Shear thinning, shear thickening behaviors described by power law, Carreau-Yasuda models etc. based on experiments and/or empirical relations are permissible within the framework of the theory presented here.

6. An important point to note is that *the material coefficients in the final form of the constitutive equations are defined using the configuration at time $t = t_n$ whereas the constitutive equations hold for the current configuration at time $t = t_{n+1}$*. This of course is a consequence of the Taylor series expansion of the coefficients in the linear combination about the configuration at time $t = t_n$. In the currently used models in the published works [22, 38, 39] for variable material coefficients, the coefficients are expressed as functions of the unknown deformation field in the current configuration at time $t = t_{n+1}$. This is obviously not supported by the derivations of the constitutive theories presented here in sections 5.4.1 and 5.4.2.

5.4.4 Special forms of rate constitutive theories for compressible thermoviscoelastic fluids

The general theory presented in sections 5.4.1 and 5.4.2 are specialized in the following sections. In section 5.5 we consider rate constitutive equations of order one in deviatoric Cauchy stress and strain rates, i.e., $m = 1$ and $n = 1$. This derivation forms the basis for Maxwell model and Giesekus model. We also consider rate constitutive equations of order one in deviatoric Cauchy stress and of order two in strain rate, i.e., $m = 1$ and $n = 2$. This derivation is presented in section 5.6 and forms the basis for Oldroyd-B model. We drop $\bar{\boldsymbol{g}}$ from the argument tensors as commonly done for the widely used constitutive models for polymeric fluids. However, inclusion of $\bar{\boldsymbol{g}}$ as argument tensor presents no special difficulty except that the number of combined generators and invariants increase. We consider the fluid to be compressible and specialize the results for the incompressible case in section 5.7.

5.5 Rate constitutive theory of orders *m*=1 and *n*=1 for the deviatoric stress tensor and heat vector: compressible thermoviscoelastic fluids

For the rate constitutive theory of orders one ($m = 1$, $n = 1$) we have

$$[{}^{(1)}_{\;d}\bar{\sigma}] = [{}^{(1)}_{\;d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(0)}_{\;d}\bar{\sigma}]\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \tag{5.16}$$

$$ {}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [{}^{(0)}_{\;d}\bar{\sigma}]\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{5.17}$$

Constitutive theory of orders *m*=1 and *n*=1 for $[{}^{(1)}_{\;d}\bar{\sigma}]$

Let $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \ldots, 12$ (see references [3–21]) be the combined generators of the argument tensors $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are symmetric tensors of rank two, and let ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \ldots, 16$

(see references [3–21]) be the combined invariants of the same argument tensors. Then, we can express $[{}_d\bar{\sigma}^{(1)}]$ as a linear combination of the generators $[{}^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \dots, 12$ and the identity tensor $[I]$ in the current configuration at time $t = t_{n+1}$.

$$[{}^{(1)}_{\;d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + \sum_{i=1}^{12} {}^{\sigma}\alpha^i[{}^{\sigma}\underset{\sim}{G}^i] \tag{5.18}$$

in which the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 12$ in (5.18) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 16$ in the current configuration at time $t = t_{n+1}$. To determine the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 12$ in (5.18) we consider the Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 12$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 16$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\Big|_{t_n} + \sum_{j=1}^{16} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\Bigg|_{t_n} \big(({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \; ; \; i = 0, 1, \dots, 12 \tag{5.19}$$

We note that ${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{q\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, 16$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \dots, 12$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, 16$ but ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}\, ,\, \bar{\theta}_{t_n}\, ,\, ({}^{q\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, 16\, ,\, \bar{\theta}_{t_{n+1}}\, ,$ $({}^{q\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \dots, 16\big)$; $i = 0, 1, \dots, 12$ in (5.19).

By substituting (5.19) in (5.18), we obtain the most general form of the rate constitutive theory for $[{}^{(1)}_{\;d}\bar{\sigma}]$ of orders 1 ($m = 1$, $n = 1$) for compressible thermoviscoelastic fluids. This theory contains too many material coefficients. Its simplifications leading to Maxwell model and Giesekus model are considered in the following sections. We follow remarks in section 5.4.3 to determine material coefficients in the final expression for $[{}^{(1)}_{\;d}\bar{\sigma}]$ and to obtain rate theories in contravariant basis, covariant basis and using Jaumann rates.

Constitutive theory of orders *m*=1 and *n*=1 for ${}^{(0)}\bar{\boldsymbol{q}}$

Let $\{{}^{q}\underset{\sim}{G}^{i}\}$; $i = 1,2,\dots,7$ (see references [3–21]) be the combined generators of argument tensors $[{}^{(0)}_{\ d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one. Let ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1,2,\dots,16$ (see references [3–21]) be the combined invariants of the same argument tensors. Then for the current configuration at time $t = t_{n+1}$

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{7} {}^{q}\alpha^{i}\{{}^{q}\underset{\sim}{G}^{i}\} \tag{5.20}$$

in which the coefficients ${}^{q}\alpha^{i}$; $i = 1,2,\dots,7$ are functions of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^{j}; j = 1,2,\dots,16$ in the current configuration at time $t = t_{n+1}$. To determine the coefficients ${}^{q}\alpha^{i}$; $i = 1,2,\dots,7$ (in the current configuration at time $t = t_{n+1}$) in (5.20), we consider Taylor series expansion of each ${}^{q}\alpha^{i}$; $i = 1,2,\dots,7$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1,2,\dots,16$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$$ {}^{q}\alpha^{i} = {}^{q}\alpha^{i}\Big|_{t_n} + \sum_{j=1}^{16} \frac{\partial({}^{q}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\Bigg|_{t_n}\Big(({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}\Big) + \frac{\partial({}^{q}\alpha^{i})}{\partial\bar{\theta}}\Bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\ ;\ i = 1,2,\dots,7 \tag{5.21}$$

We note that ${}^{q}\alpha^{i}\big|_{t_n}$, $\frac{\partial({}^{q}\alpha^{i})}{\partial({}^{q\sigma}\underset{\sim}{I}^{j})}\big|_{t_n}$; $j = 1,2,\dots,16$ and $\frac{\partial({}^{q}\alpha^{i})}{\partial\bar{\theta}}\big|_{t_n}$; $i = 1,2,\dots,7$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}$; $j = 1,2,\dots,16$ but ${}^{q}\alpha^{i} = {}^{q}\alpha^{i}\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{q\sigma}\underset{\sim}{I}^{j})_{t_n}; j = 1,2,\dots,16\ ,\ \bar{\theta}_{t_{n+1}}\ ,$ $({}^{q\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}; j = 1,2,\dots,16\big)$; $1,2,\dots,7$ in (5.21).

By substituting (5.21) in (5.20), , we obtain the general form of the rate constitutive theory for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ of orders 1 ($m = 1$, $n = 1$) for compressible thermoviscoelastic fluids. As in the case of $[{}^{(1)}_{\ d}\bar{\sigma}]$, this theory also contains too many material coefficients. The simplifications of this theory are considered in the following sections. In this case also, we follow remarks in section 5.4.3 to determine material coefficients and for obtaining specific forms of the rate theories in the desired basis.

5.5.1 Further assumptions and simplifications

In order to derive Maxwell model and Giesekus model from the rate constitutive theory of orders $m = 1$ and $n = 1$ for the deviatoric Cauchy stress tensor and the constitutive theory for the heat vector, we make further assumptions in (5.16) and (5.17). We assume that $[{}^{(1)}_{\,d}\bar{\sigma}]$ does not depend on $\bar{\boldsymbol{g}}$ hence we can eliminate $\bar{\boldsymbol{g}}$ as an argument tensor from (5.16). We also assume that the heat vector only depend upon $\bar{\boldsymbol{g}}$, $\bar{\rho}$ and $\bar{\theta}$ in (5.17), thus we can eliminate $[{}^{(0)}_{\,d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ from the arguments of the heat vector in (5.17).

$$[{}^{(1)}_{\,d}\bar{\sigma}] = [{}^{(1)}_{\,d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(0)}_{\,d}\bar{\sigma}]\ ,\ [{}^{(1)}\gamma]\ ,\ \bar{\theta})] \tag{5.22}$$

$$^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{5.23}$$

(a) Constitutive theory for $[{}^{(1)}_{\,d}\bar{\sigma}]$

The development of the constitutive theory in this case requires: (1) combined generators of $[{}^{(0)}_{\,d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ (both symmetric tensors of rank two) that are also symmetric tensors of rank two due to the fact that $[{}^{(1)}_{\,d}\bar{\sigma}]$ is a symmetric tensor of rank two (2) combined invariants of the tensors $[{}^{(0)}_{\,d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$. These are listed in tables 5.1 and 5.2 [3–21].

***Remarks*:**

1. We note that the invariants listed in table 5.2 under (2) marked (a) need not be included due to the fact that

$$\mathrm{tr}\big([{}^{(0)}_{\,d}\bar{\sigma}][{}^{(1)}\gamma] + [{}^{(1)}\gamma][{}^{(0)}_{\,d}\bar{\sigma}]\big) + \mathrm{tr}\big([{}^{(0)}_{\,d}\bar{\sigma}][{}^{(1)}\gamma] - [{}^{(1)}\gamma][{}^{(0)}_{\,d}\bar{\sigma}]\big) = 2\,\mathrm{tr}\big([{}^{(0)}_{\,d}\bar{\sigma}][{}^{(1)}\gamma]\big)$$

which is same as ${}^{\sigma}\underset{\sim}{I}^{7}$ (except for the factor 2 which is of no consequence).

2. In many published works (a) are also included in the list of invariants in addition to ${}^{q\sigma}\underset{\sim}{I}^{7}$

Table 5.1: Combined generators for $[{}^{(1)}_{d}\bar{\sigma}]$: $m = 1$, $n = 1$; first order rate theory

	Arguments	**Generators**
(1)	none	$[I]$
(2)	one at a time $\big($ including (1) $\big)$	
	$[{}^{(0)}_{d}\bar{\sigma}]$	$[{}^{\sigma}\underset{\sim}{G}^1] = [{}^{(0)}_{d}\bar{\sigma}]$; $[{}^{\sigma}\underset{\sim}{G}^2] = [{}^{(0)}_{d}\bar{\sigma}]^2$
	$[{}^{(1)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^3] = [{}^{(1)}\gamma]$; $[{}^{\sigma}\underset{\sim}{G}^4] = [{}^{(1)}\gamma]^2$
(3)	two at a time $\big($ including (1) and (2) $\big)$	
	$[{}^{(0)}_{d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$	$[{}^{\sigma}\underset{\sim}{G}^5] = [{}^{(0)}_{d}\bar{\sigma}][{}^{(1)}\gamma] + [{}^{(1)}\gamma][{}^{(0)}_{d}\bar{\sigma}]$
		$[{}^{\sigma}\underset{\sim}{G}^6] = [{}^{(0)}_{d}\bar{\sigma}]^2[{}^{(1)}\gamma] + [{}^{(1)}\gamma][{}^{(0)}_{d}\bar{\sigma}]^2$
		$[{}^{\sigma}\underset{\sim}{G}^7] = [{}^{(0)}_{d}\bar{\sigma}][{}^{(1)}\gamma]^2 + [{}^{(1)}\gamma]^2[{}^{(0)}_{d}\bar{\sigma}]$

which is redundant.

Using the generators in table 5.1 we can express $[{}^{(1)}_{d}\bar{\sigma}]$ as a linear combination of $[I]$ and the combined generators $[{}^{\sigma}\underset{\sim}{G}^i]$; $i = 1, 2, \dots, 7$. Thus, we can write the following in the current configuration at time $t = t_{n+1}$.

$$[{}^{(1)}_{d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + \sum_{i=1}^{7} {}^{\sigma}\alpha^i[{}^{\sigma}\underset{\sim}{G}^i] \tag{5.24}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 7$ are functions of the combined invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$, density $\bar{\rho}$ and temperature $\bar{\theta}$. The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 7$ are determined by using Taylor series expansion for each ${}^{\sigma}\alpha^i$ about the configuration at time $t = t_n$ and only retaining up to linear terms in the combined invariants ${}^{\sigma}\underset{\sim}{I}^{j}$ and temperature $\bar{\theta}$.

$$
{}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\Big|_{t_n} + \sum_{j=1}^{10} \frac{\partial({}^{\sigma}\alpha^{i})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\Bigg|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^{i})}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ ; \ i = 0,1,\dots,7 \quad (5.25)
$$

We note that ${}^{\sigma}\alpha^{i}\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^{i})}{\partial({}^{\sigma}\underset{\sim}{I}^{j})}\big|_{t_n}$; $j = 1,2,\dots,10$ and $\frac{\partial({}^{\sigma}\alpha^{i})}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0,1,\dots,7$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}$; $j = 1,2,\dots,10$ but ${}^{\sigma}\alpha^{i} = {}^{\sigma}\alpha^{i}\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}; j = 1,2,\dots,10\ ,\ \bar{\theta}_{t_{n+1}}\ ,$ $({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}}; j = 1,2,\dots,10\big)$; $i = 0,1,\dots,7$ in (5.25).

By substituting (5.25) in (5.24), we obtain the rate constitutive theory for $[{}^{(1)}_{\ d}\bar{\sigma}]$ based on the argument tensors in (5.22). We note that this expression for $[{}^{(1)}_{\ d}\bar{\sigma}]$ contains all the combined generators and invariants of the argument tensors listed in tables 5.1 and 5.2 and is a non-linear relationship in $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ but it is a first order rate theory ($m = 1$ and $n = 1$).

This rate theory still contains many material coefficients but its further simplifications form the basis for Maxwell and Giesekus constitutive models.

(b) Constitutive theory for ${}^{(0)}\bar{\boldsymbol{q}}$

Consider (5.23) for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$. In this case, the generators of ${}^{(0)}\bar{\boldsymbol{q}}$ that are tensors of rank one are given by $\bar{\boldsymbol{g}}$ only. Also, in this case the only invariant is ${}^{q}\underset{\sim}{I} = \bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$. Thus, we can write the following in the current configuration at time $t = t_{n+1}$.

$$
{}^{(0)}\bar{\boldsymbol{q}} = -{}^{q}\alpha\, \bar{\boldsymbol{g}} \quad (5.26)
$$

The coefficient ${}^{q}\alpha$ in (5.26) is a function of $\bar{\rho}$, $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}$ in the current configuration at time $t = t_{n+1}$ i.e., $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and $({}^{q}\underset{\sim}{I})_{t_{n+1}}$. To determine the coefficient ${}^{q}\alpha$ in (5.26) related to the current configuration at time $t = t_{n+1}$, we consider Taylor series expansion of ${}^{q}\alpha$ about the configuration

Table 5.2: Combined invariants for $[{}^{(1)}_{d}\bar{\sigma}]$: $m = 1$, $n = 1$; first order rate theory

	Arguments	**Invariants**
(1)	one at a time	
	$[{}^{(0)}_{d}\bar{\sigma}]$	${}^{\sigma}\underset{\sim}{I}^1 = \text{tr}([{}^{(0)}_{d}\bar{\sigma}])$; ${}^{\sigma}\underset{\sim}{I}^2 = \text{tr}([{}^{(0)}_{d}\bar{\sigma}]^2)$ ${}^{\sigma}\underset{\sim}{I}^3 = \text{tr}([{}^{(0)}_{d}\bar{\sigma}]^3)$
	$[{}^{(1)}\gamma]$	${}^{\sigma}\underset{\sim}{I}^4 = \text{tr}([{}^{(1)}\gamma])$; ${}^{\sigma}\underset{\sim}{I}^5 = \text{tr}([{}^{(1)}\gamma]^2)$ ${}^{\sigma}\underset{\sim}{I}^6 = \text{tr}([{}^{(1)}\gamma]^3)$
(2)	two at a time (including (1))	
	$[{}^{(0)}_{d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$	${}^{\sigma}\underset{\sim}{I}^7 = \text{tr}([{}^{(0)}_{d}\bar{\sigma}][{}^{(1)}\gamma])$; ${}^{\sigma}\underset{\sim}{I}^8 = \text{tr}([{}^{(0)}_{d}\bar{\sigma}]^2[{}^{(1)}\gamma])$ ${}^{\sigma}\underset{\sim}{I}^9 = \text{tr}([{}^{(0)}_{d}\bar{\sigma}][{}^{(1)}\gamma]^2)$; ${}^{\sigma}\underset{\sim}{I}^{10} = \text{tr}([{}^{(0)}_{d}\bar{\sigma}]^2[{}^{(1)}\gamma]^2)$ (a) $\boxed{\begin{array}{l} {}^{\sigma}\underset{\sim}{I} = \text{tr}([{}^{(0)}_{d}\bar{\sigma}][{}^{(1)}\gamma] + [{}^{(1)}\gamma][{}^{(0)}_{d}\bar{\sigma}]) \\ {}^{\sigma}\underset{\sim}{I} = \text{tr}([{}^{(0)}_{d}\bar{\sigma}][{}^{(1)}\gamma] - [{}^{(1)}\gamma][{}^{(0)}_{d}\bar{\sigma}]) \end{array}}$

at time $t = t_n$ in $\bar{\theta}$ and ${}^{q}\underset{\sim}{I}$ and retain only up to linear terms in $\bar{\theta}$ and the invariant.

$$ {}^{q}\alpha = {}^{q}\alpha\Big|_{t_n} + \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\Bigg|_{t_n} \big((\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}} - (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n} \big) + \frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \tag{5.27} $$

${}^{q}\alpha\big|_{t_n}$, $\frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\big|_{t_n}$ and $\frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\big|_{t_n}$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}$, however ${}^{q}\alpha = {}^{q}\alpha\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}\ ,\ \bar{\theta}_{t_{n+1}}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}}\big)$ in (5.27). Substituting from (5.27) into (5.26)

$$ {}^{(0)}\bar{\boldsymbol{q}} = -{}^{q}\alpha\Big|_{t_n}\bar{\boldsymbol{g}} - \frac{\partial({}^{q}\alpha)}{\partial({}^{q}\underset{\sim}{I})}\Bigg|_{t_n} \big((\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_{n+1}} - (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n} \big)\,\bar{\boldsymbol{g}} - \frac{\partial({}^{q}\alpha)}{\partial\bar{\theta}}\Bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\,\bar{\boldsymbol{g}} \tag{5.28} $$

We note that if there is a uniform temperature change between the configurations at times t_n and t_{n+1}, then $\bar{\boldsymbol{g}} = 0$ and hence ${}^{(0)}\bar{\boldsymbol{q}}$ must be zero. This condition is satisfied by (5.28). We note that all quantities at time t_n are known as these correspond to a configuration for which the deformation field is known. We collect terms and define material coefficients and others. Let

$$
\begin{aligned}
k_{t_n} &= {}^q\alpha\Big|_{t_n} - \frac{\partial({}^q\alpha)}{\partial({}^q\underset{\sim}{I})}\bigg|_{t_n} (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n} \\
(k_1)_{t_n} &= \frac{\partial({}^q\alpha)}{\partial({}^q\underset{\sim}{I})}\bigg|_{t_n} \\
(k_2)_{t_n} &= \frac{\partial({}^q\alpha)}{\partial\bar{\theta}}\bigg|_{t_n}
\end{aligned}
\tag{5.29}
$$

then (5.28) becomes (we drop the subscript t_{n+1} since it is understood it represents the current configuration)

$$
{}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} - (k_1)_{t_n}(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})\,\bar{\boldsymbol{g}} - (k_2)_{t_n}(\bar{\theta}-\bar{\theta}_{t_n})\,\bar{\boldsymbol{g}}
\tag{5.30}
$$

We note that $k_{t_n} = k(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$, $(k_1)_{t_n} = k_1(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$ and $(k_2)_{t_n} = k_2(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n})$. Equation (5.30) is the most general form of the constitutive equation for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ based on (5.23). If we neglect the last term in (5.30) then

$$
{}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}} - (k_1)_{t_n}(\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})\,\bar{\boldsymbol{g}}
\tag{5.31}
$$

and if we neglect infinitesimals of order two and higher in the components of $\bar{\boldsymbol{g}}$, then

$$
{}^{(0)}\bar{\boldsymbol{q}} = -k_{t_n}\,\bar{\boldsymbol{g}}
\tag{5.32}
$$

$$
\text{or}\qquad {}^{(0)}\bar{\boldsymbol{q}} = -k\,\bar{\boldsymbol{g}} = -k[I]\,\bar{\boldsymbol{g}} = -[K]\,\bar{\boldsymbol{g}}
\tag{5.33}
$$

in which k is *thermal conductivity* and $[K]$ is the *diagonal thermal conductivity matrix*. We note that

$$
k = k_{t_n} = k\big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ (\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}})_{t_n}\big)
\tag{5.34}
$$

Equation (5.33) is the *standard Fourier heat conduction law with variable thermal conductivity*. Based on 5.34, the thermal conductivity can be a function of density, temperature and the first invariant of $\bar{\boldsymbol{g}}$ i.e., $\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}}$. Thus, we can use experimental and/or empirical data for thermal conductivity as a function of density, temperature and $\bar{\boldsymbol{g}}\cdot\bar{\boldsymbol{g}}$ during the evolution, keeping in mind that 5.34

only holds for $t = t_n$ where as ${}^{(0)}\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{g}}$ in (5.33) are in the current configuration at time $t = t_n$. This is obviously a consequence of Taylor series expansion of ${}^{(0)}\bar{\boldsymbol{q}}$ about the configuration at time $t = t_n$. Power law, Sutherland law [22, 38, 39] are examples of temperature dependent thermal conductivities.

In currently published works [22, 38, 39] the thermal conductivity is generally expressed as a function of the unknown deformation or state in the configuration at time $t = t_{n+1}$ i.e., instead of k_{t_n} defined by 5.34, it is replaced by

$$k = k_{t_{n+1}} = k\big(\bar{\rho}_{t_{n+1}}\,,\ \bar{\theta}_{t_{n+1}}\,,\ (\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}})_{t_{n+1}}\big) \tag{5.35}$$

This obviously makes k a function of unknown $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}} \cdot \bar{\boldsymbol{g}}$ in the current configuration. When the two configurations at times $t = t_n$ and $t = t_{n+1}$ are in close proximity of each other in terms of deformation field, replacing k in (5.33) using 5.35 may be justified, but it is not supported by the derivation of the constitutive theory presented here.

We note that this constitutive theory ((5.30) and (5.33)) for ${}^{(0)}\bar{\boldsymbol{q}}$ is independent of the basis i.e.

$${}^{(0)}\bar{\boldsymbol{q}} = \bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}_{(0)} = {}^{(0)}\bar{\boldsymbol{q}}^{J} = \bar{\boldsymbol{q}} \tag{5.36}$$

5.5.2 Maxwell constitutive model for deviatoric Cauchy stress tensor

The simplified first order ($m = 1$, $n = 1$) rate constitutive theory presented in section 5.5.1 can be shown to yield Maxwell constitutive model used for dilute polymeric fluids upon further assumptions and simplifications. We present details in this section. Maxwell constitutive model is a *linear viscoelastic model*, hence $[{}^{(1)}_{\ d}\bar{\sigma}]$ must only be a function of the generators $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$

in addition to $\bar{\rho}$ and $\bar{\theta}$. Therefore (5.24) reduces

$$[{}^{(1)}_{\ d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{(0)}_{\ d}\bar{\sigma}] + {}^{\sigma}\alpha^2[{}^{(1)}\gamma] \tag{5.37}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ depend upon $\bar{\rho}$, $\bar{\theta}$ and the combined invariants of $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ in the current configuration at time $t = t_{n+1}$. These are listed in table 5.2 (${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$). To determine the coefficients in ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ in (5.37), we consider Taylor series expansions of ${}^{\sigma}\alpha^i$; $i = 0, 1, 2$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{10} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ \ ; \ \ i = 0, 1, 2 \tag{5.38}$$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, 10$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, 2$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, 10$, however ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n} \ , \ \bar{\theta}_{t_n} \ , \ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, 10 \ , \ \bar{\theta}_{t_{n+1}} \ , \ ({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}$; $j = 1, 2, \dots, 10\big)$; $i = 0, 1, 2$ in (5.38). If we let ${}^{\sigma}\alpha^i{}_{,j} = \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}$; $j = 1, 2, \dots, 10$, then (5.38) can be written as

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^i{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ \ ; \ \ i = 0, 1, 2 \tag{5.39}$$

Substituting from (5.39) in (5.37), we obtain the most general expression for the constitutive theory for $[{}^{(1)}_{\ d}\bar{\sigma}]$ based on the choice of generators in (5.37) and invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$ listed in table 5.2.

$$\begin{aligned}
[{}^{(1)}_{\ d}\bar{\sigma}] = &\Big({}^{\sigma}\alpha^0\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^0{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] + \\
&\Big({}^{\sigma}\alpha^1\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^1{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(0)}_{\ d}\bar{\sigma}] + \\
&\Big({}^{\sigma}\alpha^2\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^2{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma]
\end{aligned} \tag{5.40}$$

(a) Further assumptions and simplifications

We make the following assumptions based on the fact that the Maxwell model is a linear viscoelastic model:

(i) We delete the terms containing products of the generators $[{}^{(0)}_{d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ with the invariants ${}^{\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \ldots, 10$ in the current configuration.

(ii) We also delete the terms containing the products of the generators $[{}^{(0)}_{d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ with $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ in the current configuration. This gives

$$\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] = & \Big({}^{\sigma}\alpha^0\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^0,_j)\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \big) + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \Big) [I] \\
& + \Big({}^{\sigma}\alpha^1\big|_{t_n} - \sum_{j=1}^{10} ({}^{\sigma}\alpha^1,_j)\big|_{t_n} ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \Big) [{}^{(0)}_{d}\bar{\sigma}] \\
& + \Big({}^{\sigma}\alpha^2\big|_{t_n} - \sum_{j=1}^{10} ({}^{\sigma}\alpha^2,_j)\big|_{t_n} ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \Big) [{}^{(1)}\gamma]
\end{aligned} \tag{5.41}$$

We collect terms and define material coefficients and others. Let

$$\begin{aligned}
{}^{\sigma}b^0 = & \ {}^{\sigma}\alpha^0\big|_{t_n} - \sum_{j=1}^{10} ({}^{\sigma}\alpha^0,_j)\big|_{t_n} ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \\
{}^{\sigma}b^1_j = & \ ({}^{\sigma}\alpha^0,_j)\big|_{t_n} \qquad ; \qquad j = 1, 2, \ldots, 10 \\
{}^{\sigma}b^2 = & \ {}^{\sigma}\alpha^1\big|_{t_n} - \sum_{j=1}^{10} ({}^{\sigma}\alpha^1,_j)\big|_{t_n} ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \\
{}^{\sigma}b^3 = & \ {}^{\sigma}\alpha^2\big|_{t_n} - \sum_{j=1}^{10} ({}^{\sigma}\alpha^2,_j)\big|_{t_n} ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}
\end{aligned} \tag{5.42}$$

Then we can write

$$\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] = & {}^{\sigma}b^0[I] + \sum_{j=1}^{10} {}^{\sigma}b^1_j ({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} [I] + {}^{\sigma}b^2 [{}^{(0)}_{d}\bar{\sigma}] \\
& + {}^{\sigma}b^3 [{}^{(1)}\gamma] + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) [I]
\end{aligned} \tag{5.43}$$

We note that the coefficients ${}^{\sigma}b^0$, ${}^{\sigma}b^1_j$; $j = 1, 2, \ldots, 10$, ${}^{\sigma}b^2$, ${}^{\sigma}b^3$ and $\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\big|_{t_n}$ are material coefficients defined in the configuration at time $t = t_n$ and are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and

$({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, 10.$

(iii) In (5.43) we can consider one more simplification. We only retain invariants $({}^{\sigma}\underset{\sim}{I}^1)_{t_{n+1}} = (i_{{}^{(0)}_{d}\bar{\sigma}})_{t_{n+1}} = \mathrm{tr}([{}^{(0)}_{d}\bar{\sigma}])$ and $({}^{\sigma}\underset{\sim}{I}^4)_{t_{n+1}} = (i_{{}^{(1)}\gamma})_{t_{n+1}} = \mathrm{tr}([{}^{(1)}\gamma])$. All other invariants in (5.43) can be removed because the Maxwell model is a linear viscoelastic model. To be more precise, in this constitutive theory, $[{}^{(1)}_{d}\bar{\sigma}]$ is a linear function of the components of the generators $[{}^{(0)}_{d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$. Thus (5.43) reduces to the following:

$$\begin{aligned}[{}^{(1)}_{d}\bar{\sigma}] =& {}^{\sigma}b^0[I] + {}^{\sigma}b_1^1\mathrm{tr}([{}^{(0)}_{d}\bar{\sigma}])[I] + {}^{\sigma}b_4^1\mathrm{tr}([{}^{(1)}\gamma])[I] + {}^{\sigma}b^2[{}^{(0)}_{d}\bar{\sigma}] + {}^{\sigma}b^3[{}^{(1)}\gamma] \\ &+ \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]\end{aligned} \tag{5.44}$$

which can be written as

$$\begin{aligned}[{}^{(0)}_{d}\bar{\sigma}] + \Big(-\frac{1}{{}^{\sigma}b^2}\Big)[{}^{(1)}_{d}\bar{\sigma}] =& \Big(-\frac{{}^{\sigma}b^0}{{}^{\sigma}b^2}\Big)[I] + \Big(-\frac{{}^{\sigma}b^3}{{}^{\sigma}b^2}\Big)[{}^{(1)}\gamma] + \Big(-\frac{{}^{\sigma}b_4^1}{{}^{\sigma}b^2}\Big)\mathrm{tr}([{}^{(1)}\gamma])[I] \\ &+ \Big(-\frac{{}^{\sigma}b_1^1}{{}^{\sigma}b^2}\Big)\mathrm{tr}([{}^{(0)}_{d}\bar{\sigma}])[I] \\ &- \Big(\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}\frac{1}{{}^{\sigma}b^2}\Big)(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]\end{aligned} \tag{5.45}$$

We introduce mew notations for the material coefficients to conform to the standard notations used in the literature. Let

$$\begin{aligned}\Big(-\frac{1}{{}^{\sigma}b^2}\Big) =&\; \lambda_{t_n} = \lambda\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\ \Big(-\frac{{}^{\sigma}b^0}{{}^{\sigma}b^2}\Big) =&\; \bar{\sigma}_0\big|_{t_n} = \bar{\sigma}_0\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\ \Big(-\frac{{}^{\sigma}b^3}{{}^{\sigma}b^2}\Big) =&\; 2\eta_{t_n} = 2\eta\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\ \Big(-\frac{{}^{\sigma}b_4^1}{{}^{\sigma}b^2}\Big) =&\; \kappa_{t_n} = \kappa\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\ \Big(-\frac{{}^{\sigma}b_1^1}{{}^{\sigma}b^2}\Big) =&\; {}^{1}\kappa_{t_n} = {}^{1}\kappa\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\ \Big(\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}\frac{1}{{}^{\sigma}b^2}\Big) =&\; (\alpha_{tm})_{t_n} = \alpha_{tm}\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big)\end{aligned} \tag{5.46}$$

then (5.45) can be written as

$$
\begin{aligned}
[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] =& \bar{\sigma}_0\big|_{t_n}[I] + 2\eta_{t_n}[{}^{(1)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I] \\
&+ {}^{1}\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])[I] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.47}
$$

in which $[{}^{(1)}_{\ d}\bar{\sigma}]$ is a linear function of the components of the generators $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$.

(iv) To derive the standard Maxwell model [22], we further assume that: (a) the initial stress field $\bar{\sigma}_0\big|_{t_n}[I]$ is zero (b) the stress field due to thermal expansion and contraction is neglected and (c) we delete the term containing the material coefficient ${}^{1}\kappa_{t_n}$.

$$
[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] = 2\eta_{t_n}[{}^{(1)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I] \tag{5.48}
$$

λ_{t_n} is called *relaxation time*, η_{t_n} is *viscosity*, κ_{t_n} is *second viscosity* and $(\alpha_{tm})_{t_n}$ is *thermal modulus*. (5.48) is the standard form of the *Maxwell constitutive model for compressible thermoviscoelastic fluids* in which λ_{t_n}, η_{t_n} and κ_{t_n} are variable transport properties.

(b) Remarks:

1. First, we note that the coefficients λ_{t_n}, η_{t_n}, κ_{t_n}, $(\alpha_{tm})_{t_n}$ are defined in the configuration at time $t = t_n$ for which deformation is known, whereas all other quantities in (5.47) and (5.48) are in the current configuration at time $t = t_{n+1}$. This is a consequence of Taylor series expansion of the coefficients about the configuration at time $t = t_n$.

2. Based on (5.46), λ_{t_n}, η_{t_n} and κ_{t_n} can be deformation dependent during evolution (keeping 1. in mind) permitting experimental and/or empirical description of λ_{t_n}, η_{t_n} and κ_{t_n} using their arguments given in (5.46). Thus, power law, Sutherland law etc. for λ_{t_n}, η_{t_n} and κ_{t_n} dependent on $\bar{\rho}_{t_n}$ and $\bar{\theta}_{t_n}$ are valid. Dependence of λ_{t_n}, η_{t_n} and κ_{t_n} on the combined invariants of $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ allows us to represent variable shear thinning and shear thickening behaviors during the evolution. Power law, Carreau-Yasuda models etc. are valid based on (5.46)

(contrary to the belief that these models do not have continuum mechanics foundation [22]).

3. By replacing $([{}^{(1)}_{\;d}\bar{\sigma}],\ [{}^{(0)}_{\;d}\bar{\sigma}],\ [{}^{(1)}\gamma])$ with $([{}_{d}\bar{\sigma}^{(1)}],\ [{}_{d}\bar{\sigma}^{(0)}],\ [\gamma^{(1)}])$, $([{}_{d}\bar{\sigma}_{(1)}],\ [{}_{d}\bar{\sigma}_{(0)}],\ [\gamma_{(1)}])$ and $([{}^{(1)}_{\;d}\bar{\sigma}^{J}],\ [{}^{(0)}_{\;d}\bar{\sigma}^{J}],\ [{}^{(1)}\gamma^{J}])$ in (5.47) and (5.48) (including coefficients), we obtain the Maxwell model that correspond to contravariant basis (*upper convected*), covariant basis (*lower convected*) and Jaumann rates. It is rather obvious that when the deformation is finite, all three rate constitutive equations will represent different physics, upper convected case being in most agreement with the physics of finite deformation [30].

4. If we assume that the configuration at times t_n and t_{n+1} are in close proximity of each other, then in (5.46) all coefficients can be expressed in terms of their arguments at time $t = t_{n+1}$ by using (5.46) and replacing t_n with t_{n+1}. This is what is done in currently used models for variable material coefficients [22, 38, 39]. This is obviously not supported by the derivation presented here. In this case the material coefficients are functions of the unknown deformation field in the current configuration.

5. The constitutive theories for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ have already been presented. Generally, Fourier heat conduction law (section 5.5.1) is commonly used in the majority of the published work.

5.5.3 Giesekus constitutive model for deviatoric Cauchy stress tensor

The general derivation presented for the constitutive theory of order one ($m = 1$, $n = 1$) for deviatoric Cauchy stress in section 5.5.1 also forms the basis for deriving the Giesekus constitutive model used for polymer melts (dense polymers). Giesekus constitutive model is a *non-linear viscoelastic model*. In the derivation of the Giesekus constitutive model, we begin by considering the combined generators of $[{}^{(0)}_{\;d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$ only up to quadratics. Therefore (5.24) reduces

$$\begin{aligned}
[{}^{(1)}_{\;d}\bar{\sigma}] =& {}^{\sigma}\alpha^{0}[I] + {}^{\sigma}\alpha^{1}[{}^{(0)}_{\;d}\bar{\sigma}] + {}^{\sigma}\alpha^{2}[{}^{(1)}\gamma] + {}^{\sigma}\alpha^{3}[{}^{(0)}_{\;d}\bar{\sigma}]^{2} \\
&+ {}^{\sigma}\alpha^{4}[{}^{(1)}\gamma]^{2} + {}^{\sigma}\alpha^{5}([{}^{(0)}_{\;d}\bar{\sigma}][{}^{(1)}\gamma] + [{}^{(1)}\gamma][{}^{(0)}_{\;d}\bar{\sigma}])
\end{aligned} \tag{5.49}$$

We further neglect the last two generators in (5.49)

$$[{}^{(1)}_{d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{(0)}_{d}\bar{\sigma}] + {}^{\sigma}\alpha^2[{}^{(1)}\gamma] + {}^{\sigma}\alpha^3[{}^{(0)}_{d}\bar{\sigma}]^2 \tag{5.50}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 3$ depend upon $\bar{\rho}$, $\bar{\theta}$ and the combined invariants of $[{}^{(0)}_{d}\bar{\sigma}]$ and $[{}^{(1)}\gamma]$, i.e., ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$, in the current configuration at time $t = t_{n+1}$. These are listed in table 5.2. To determine the coefficients in ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 3$ in (5.50), we consider Taylor series expansions of ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, 3$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{10} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \;\; ; \;\; i = 0, 1, \dots, 3 \tag{5.51}$$

Coefficients ${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \dots, 10$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \dots, 3$ are functions of $\bar{\rho}_{t_n}, \bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, 10$, however ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \dots, 10, \bar{\theta}_{t_{n+1}}, ({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \dots, 10\big)$; $i = 0, 1, \dots, 3$ in (5.51). If we let ${}^{\sigma}\alpha^i{}_{,j} = \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}$; $j = 1, 2, \dots, 10$, then (5.51) can be written as

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^i{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \;\; ; \;\; i = 0, 1, \dots, 3 \tag{5.52}$$

Substituting from (5.52) in (5.50), we obtain the most general expression for the constitutive theory for $[{}^{(1)}_{d}\bar{\sigma}]$ based on the choice of generators in (5.50) and invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$ listed in table 5.2.

$$\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] = &\Big({}^{\sigma}\alpha^0\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^0{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] + \\
&\Big({}^{\sigma}\alpha^1\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^1{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^1)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(0)}_{d}\bar{\sigma}] + \\
&\Big({}^{\sigma}\alpha^2\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^2{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^2)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma] + \\
&\Big({}^{\sigma}\alpha^3\big|_{t_n} + \sum_{j=1}^{10} ({}^{\sigma}\alpha^3{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^3)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(0)}_{d}\bar{\sigma}]^2
\end{aligned} \tag{5.53}$$

(a) Further assumptions and simplifications

We make the following assumptions to derive the Giesekus model, which is a non-linear viscoelastic model:

(i) We delete the terms containing products of the generators $[{}^{(1)}\gamma]$, $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(0)}_{\ d}\bar{\sigma}]^2$ with the invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, 10$ in the current configuration.

(ii) We also delete the terms containing the products of the generators $[{}^{(1)}\gamma]$, $[{}^{(0)}_{\ d}\bar{\sigma}]$ and $[{}^{(0)}_{\ d}\bar{\sigma}]^2$ with $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ in the current configuration. This gives

$$
\begin{aligned}
[{}^{(1)}_{\ d}\bar{\sigma}] = & \Big({}^{\sigma}\alpha^0\big|_{t_n} + \sum_{j=1}^{10}({}^{\sigma}\alpha^0,_j)\big|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] \\
& + \Big({}^{\sigma}\alpha^1\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^1,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\Big)[{}^{(0)}_{\ d}\bar{\sigma}] \\
& + \Big({}^{\sigma}\alpha^2\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^2,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\Big)[{}^{(1)}\gamma] \\
& + \Big({}^{\sigma}\alpha^3\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^3,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\Big)[{}^{(0)}_{\ d}\bar{\sigma}]^2
\end{aligned}
\tag{5.54}
$$

We collect terms and define material coefficients and others. Let

$$
\begin{aligned}
{}^{\sigma}b^0 = & \ {}^{\sigma}\alpha^0\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^0,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \\
{}^{\sigma}b^1_j = & \ ({}^{\sigma}\alpha^0,_j)\big|_{t_n} \quad ; \quad j = 1, 2, \dots, 10 \\
{}^{\sigma}b^2 = & \ {}^{\sigma}\alpha^1\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^1,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \quad ; \quad {}^{\sigma}b^3 = {}^{\sigma}\alpha^2\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^2,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n} \\
{}^{\sigma}b^4 = & \ {}^{\sigma}\alpha^3\big|_{t_n} - \sum_{j=1}^{10}({}^{\sigma}\alpha^3,_j)\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^j)_{t_n}
\end{aligned}
\tag{5.55}
$$

Then we can write

$$
\begin{aligned}
[{}^{(1)}_{\ d}\bar{\sigma}] = & {}^{\sigma}b^0[I] + \sum_{j=1}^{10}{}^{\sigma}b^1_j({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}[I] + {}^{\sigma}b^2[{}^{(0)}_{\ d}\bar{\sigma}] + {}^{\sigma}b^3[{}^{(1)}\gamma] \\
& + {}^{\sigma}b^4[{}^{(0)}_{\ d}\bar{\sigma}]^2 + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.56}
$$

We note that the coefficients ${}^{\sigma}b^0$, ${}^{\sigma}b^1_j$; $j = 1, 2, \ldots, 10$, ${}^{\sigma}b^2$, ${}^{\sigma}b^3$, ${}^{\sigma}b^4$ and $\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\Big|_{t_n}$ defined in the configuration at time $t = t_n$ and hence are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, 10$. The constitutive model (5.56) is quite general based on (5.50) and the assumptions (i) and (ii).

(iii) To derive the Giesekus constitutive model, we neglect all invariants in (5.56) except (based on table 5.2) $({}^{\sigma}\underset{\sim}{I}^1)_{t_{n+1}} = (i_{{}^{(0)}_{d}\bar{\sigma}})_{t_{n+1}} = \text{tr}([{}^{(0)}_{d}\bar{\sigma}])$ and $({}^{\sigma}\underset{\sim}{I}^4)_{t_{n+1}} = (i_{{}^{(1)}\gamma})_{t_{n+1}} = \text{tr}([{}^{(1)}\gamma])$. Thus (5.56) reduces to the following:

$$\begin{aligned}
[{}^{(1)}_{d}\bar{\sigma}] =& {}^{\sigma}b^0[I] + {}^{\sigma}b^1_1\text{tr}([{}^{(0)}_{d}\bar{\sigma}])[I] + {}^{\sigma}b^1_4\text{tr}([{}^{(1)}\gamma])[I] + {}^{\sigma}b^2[{}^{(0)}_{d}\bar{\sigma}] + {}^{\sigma}b^3[{}^{(1)}\gamma] \\
&+ {}^{\sigma}b^4[{}^{(0)}_{d}\bar{\sigma}]^2 + \frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned} \tag{5.57}$$

which can be written as

$$\begin{aligned}
[{}^{(0)}_{d}\bar{\sigma}] + \Big(-\frac{1}{{}^{\sigma}b^2}\Big)[{}^{(1)}_{d}\bar{\sigma}] =& \Big(-\frac{{}^{\sigma}b^0}{{}^{\sigma}b^2}\Big)[I] + \Big(-\frac{{}^{\sigma}b^3}{{}^{\sigma}b^2}\Big)[{}^{(1)}\gamma] + \Big(-\frac{{}^{\sigma}b^1_4}{{}^{\sigma}b^2}\Big)\text{tr}([{}^{(1)}\gamma])[I] \\
&+ \Big(-\frac{{}^{\sigma}b^1_1}{{}^{\sigma}b^2}\Big)\text{tr}([{}^{(0)}_{d}\bar{\sigma}])[I] + \Big(-\frac{{}^{\sigma}b^4}{{}^{\sigma}b^2}\Big)[{}^{(0)}_{d}\bar{\sigma}]^2 \\
&- \Big(\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}\frac{1}{{}^{\sigma}b^2}\Big)(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned} \tag{5.58}$$

We introduce mew notations for the material coefficients to conform to the standard notations used in the literature. Let

$$\begin{aligned}
\Big(-\frac{1}{{}^{\sigma}b^2}\Big) &= \lambda_{t_n} = \lambda\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\
\Big(-\frac{{}^{\sigma}b^0}{{}^{\sigma}b^2}\Big) &= \bar{\sigma}_0\big|_{t_n} = \bar{\sigma}_0\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\
\Big(-\frac{{}^{\sigma}b^3}{{}^{\sigma}b^2}\Big) &= 2\eta_{t_n} = 2\eta\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\
\Big(-\frac{{}^{\sigma}b^1_4}{{}^{\sigma}b^2}\Big) &= \kappa_{t_n} = \kappa\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\
\Big(-\frac{{}^{\sigma}b^1_1}{{}^{\sigma}b^2}\Big) &= {}^{1}\kappa_{t_n} = {}^{1}\kappa\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big) \\
\Big(\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\bigg|_{t_n}\frac{1}{{}^{\sigma}b^2}\Big) &= (\alpha_{tm})_{t_n} = \alpha_{tm}\Big(\bar{\rho}_{t_n}\,,\,\bar{\theta}_{t_n}\,,\,({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, 10\Big)
\end{aligned} \tag{5.59}$$

then (5.58) can be written as

$$\begin{aligned}[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] =&\bar{\sigma}_0\big|_{t_n}[I] + 2\eta_{t_n}[{}^{(1)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I] + {}^1\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])[I] \\ &+ \Big(-\frac{{}^{\sigma}b^4}{{}^{\sigma}b^2}\Big)[{}^{(0)}_{\ d}\bar{\sigma}]^2 - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]\end{aligned} \tag{5.60}$$

We note that each term in (5.60) has dimension of stress, thus the coefficient of $[{}^{(0)}_{\ d}\bar{\sigma}]^2$ must have dimension of (1/stress) which is same as (time/dimension of viscosity) i.e., λ_{t_n}/η_{t_n}. We choose

$$-\frac{{}^{\sigma}b^4}{{}^{\sigma}b^2} = \frac{\lambda_{t_n}}{\eta_{t_n}}\alpha \tag{5.61}$$

α being a dimensionless parameter called *mobility factor*. Therefore

$$\begin{aligned}[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] =&\bar{\sigma}_0\big|_{t_n}[I] + 2\eta_{t_n}[{}^{(1)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I] + {}^1\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])[I] \\ &+ \frac{\lambda_{t_n}}{\eta_{t_n}}\alpha[{}^{(0)}_{\ d}\bar{\sigma}]^2 - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]\end{aligned} \tag{5.62}$$

(iv) In the derivation of the standard Giesekus model [22], we further assume that: (a) the initial stress field associated with the configuration at time $t = t_n$, i.e., $\bar{\sigma}_0\big|_{t_n}[I]$, is zero (b) the stress field due to thermal expansion and contraction between the current configuration at time $t = t_{n+1}$ and the configuration at time $t = t_n$, i.e., the last term in (5.62), is neglected and (c) we further assume that ${}^1\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])$ can be neglected. Thus we obtain

$$[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] = 2\eta_{t_n}[{}^{(1)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I] + \frac{\lambda_{t_n}}{\eta_{t_n}}\alpha[{}^{(0)}_{\ d}\bar{\sigma}]^2 \tag{5.63}$$

in which λ_{t_n} is called *relaxation time*, η_{t_n} is *first viscosity*, κ_{t_n} is *second viscosity* and $(\alpha_{tm})_{t_n}$ is *thermal modulus*. (5.63) is the *Giesekus constitutive model for compressible thermoviscoelastic fluids* in which λ_{t_n}, η_{t_n}, κ_{t_n} and $(\alpha_{tm})_{t_n}$ are variable transport properties.

(b) Remarks:

1. The coefficients λ_{t_n}, η_{t_n}, κ_{t_n}, $(\alpha_{tm})_{t_n}$ are defined in the configuration at time $t = t_n$ for which deformation is known, whereas all other quantities in (5.60) and (5.63) are in the current configuration at time $t = t_{n+1}$. This is a consequence of Taylor series expansion of the coefficients about the configuration at time $t = t_n$.

2. Based on (5.59), λ_{t_n}, η_{t_n}, κ_{t_n} etc. can be deformation dependent during evolution (keeping 1. in mind) permitting experimental and/or empirical description of λ_{t_n}, η_{t_n}, κ_{t_n} etc. using their arguments given in (5.59). Thus, power law, Sutherland law etc. for λ_{t_n}, η_{t_n}, κ_{t_n} etc. dependent on $\bar{\rho}_{t_n}$ and $\bar{\theta}_{t_n}$ are valid. Dependence of λ_{t_n}, η_{t_n}, κ_{t_n} etc. on $(\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, 10$ allows us to represent variable shear thinning and shear thickening behaviors during the evolution. Power law, Carreau-Yasuda models etc. are valid based on (5.59) (contrary to the belief that these models do not have continuum mechanics foundation [22]).

3. By replacing $([{}^{(1)}_{\;d}\bar{\sigma}], [{}^{(0)}_{\;d}\bar{\sigma}], [{}^{(1)}\gamma])$ with $([{}_d\bar{\sigma}^{(1)}], [{}_d\bar{\sigma}^{(0)}], [\gamma^{(1)}])$, $([{}_d\bar{\sigma}_{(1)}], [{}_d\bar{\sigma}_{(0)}], [\gamma_{(1)}])$ and $([{}^{(1)}_{\;d}\bar{\sigma}^J], [{}^{(0)}_{\;d}\bar{\sigma}^J], [{}^{(1)}\gamma^J])$ in (5.60) and (5.63), we obtain the Giesekus model that correspond to contravariant basis (*upper convected*), covariant basis (*lower convected*) and Jaumann rates. It is rather obvious that when the deformation is finite, all three rate constitutive equations will represent different physics, upper convected case being in most agreement with the physics of finite deformation [30].

4. If we assume that the configurations at times t_n and t_{n+1} are in close proximity of each other, then in (5.59) all coefficients can be expressed in terms of their arguments at time $t = t_{n+1}$ instead of at time $t = t_n$ (as shown in (5.59)). This is what is done in currently used models for variable material coefficients [22,38,39]. This is obviously not supported by the derivation presented here. By replacing t_n with t_{n+1} in (5.59), the material coefficients become functions of the unknown deformation field in the current configuration (at $t = t_{n+1}$).

5. The constitutive theories for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ have already been presented. Generally, Fourier heat conduction law (section 5.5.1) is commonly used in the majority of the published work.

6. It is important to note that the constitutive model (5.63) uses the first convected time derivative of the deviatoric Cauchy stress tensor as a dependent variable in the development of the constitutive theory, thus (5.63) are the constitutive equations in deviatoric Cauchy stress tensor $[{}^{(0)}_{d}\bar{\sigma}]$ and its convected time derivative. This derivation is supported by the axioms and principles of continuum mechanics and the constitutive theory. In the presently used Giesekus constitutive model, this is not the case. We present details in the following.

(c) Discussion on Giesekus model presented in this chapter and model used currently

We note that the entropy inequality requires decomposition of the Cauchy stress tensor (in contra- or co-variant or Jaumann basis) into equilibrium stress and deviatoric stress. The constitutive theory for the equilibrium stress using entropy inequality results in thermodynamic pressure $\bar{p}(\bar{\rho},\bar{\theta})$ for compressible thermoviscoelastic fluids and mechanical pressure $\bar{p}(\bar{\theta})$ for incompressible case. Since the entropy inequality only requires the conversion of mechanical energy due to the deviatoric Cauchy stress to be positive but provides no mechanism for establishing the constitutive theory for it, the theory of invariants and generators is used for deriving the constitutive theory for it. The use of the deviatoric Cauchy stress tensor in the Giesekus constitutive model derived here is necessitated due to the entropy inequality. In the currently used Giesekus constitutive model for the stress tensor, the deviatoric Cauchy stress is further decomposed into solvent stress and polymer stress. If we consider contravariant basis, then

$$[{}_{d}\bar{\sigma}^{(0)}] = [{}_{d}\bar{\sigma}^{(0)}]_s + [{}_{d}\bar{\sigma}^{(0)}]_p \tag{5.64}$$

in which s and p stand for solvent and polymer. The currently used Giesekus constitutive model contains exactly the same form as presented here but uses $[{}_{d}\bar{\sigma}^{(0)}]_p$ instead of $[{}_{d}\bar{\sigma}^{(0)}]$ and is derived using Brownian motion of polymer molecules and kinetic theory [25, 51]. For the solvent stress $[{}_d\bar{\sigma}^{(0)}]_s$, Newton's law of viscosity is assumed as a constitutive theory. We note the following:

1. If we use the decomposition shown above and substitute it in the conditions resulting from the entropy inequality we still have the same restriction that the conversion of mechanical energy due to both solvent and polymer deviatoric Cauchy stress tensors be positive, but we have no mechanism for deriving constitutive theories for either one of them.

2. If we derive the Giesekus constitutive model using the theory of invariants and generators using $[{}_d\bar{\sigma}^{(0)}]_p$ as a dependent variable in the constitutive theory and if we assume Newton's law of viscosity for $[{}_d\bar{\sigma}^{(0)}]_s$, then of course we would obtain exactly the same Giesekus constitutive model as used currently. Of course, the question is "Is this permissible within the framework of the axioms of the constitutive theory and principles of continuum mechanics?".

3. Based on the axioms of the constitutive theory and the entropy inequality, $[{}_d\bar{\sigma}^{(0)}]$ is a fundamental dependent variable in the rate constitutive theory for thermoviscoelastic fluids and hence must be used as dependent variable in the derivation of the rate theory.

4. If we follow 2. i.e., if we use $[{}_d\bar{\sigma}^{(0)}]_p$ as a dependent variable in the rate theory for Giesekus constitutive model, then the constitutive theory for $[{}_d\bar{\sigma}^{(0)}]_s$ must be derivable as well from the entropy inequality (and not assuming Newton's law of viscosity for it). This is obviously not possible.

5. Thus, based on the work presented here, we conclude that the use of the deviatoric Cauchy stress tensor as a dependent variable is necessary in the derivation of the Giesekus constitutive model. This is consistent with the conditions resulting from the entropy inequality and the axioms of the constitutive theory based on continuum mechanics. Furthermore, there is no justification based on the entropy inequality for the decomposition (5.64) as $[{}_d\bar{\sigma}^{(0)}]_s$ is not derivable using the conditions resulting from the entropy inequality. The use of Newton's law of viscosity may be a good engineering assumption but it has no basis in view of

the entropy inequality and the axioms and principles of the constitutive theory in continuum mechanics.

6. It is rather obvious that the use of the Giesekus constitutive model presented in this chapter and that used currently in the mathematical models derived using conservation laws of deforming thermoviscoelastic fluids will undoubtedly produce different behaviors.

5.6 Rate constitutive theory of orders *m*=1 and *n*=2 for the deviatoric Cauchy stress tensor and heat vector: compressible thermoviscoelastic fluids

We consider rate constitutive theory of order one in stress rate and of order two in strain rate for the deviatoric Cauchy stress tensor and the heat vector. This derivation forms the basis for Oldroyd-B constitutive model.

$$[{}^{(1)}_{\;\;d}\bar{\sigma}] = [{}^{(1)}_{\;\;d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(0)}_{\;\;d}\bar{\sigma}]\ ,\ [{}^{(1)}\gamma]\ ,\ [{}^{(2)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}})] \tag{5.65}$$

$${}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ [{}^{(0)}_{\;\;d}\bar{\sigma}]\ ,\ [{}^{(1)}\gamma]\ ,\ [{}^{(2)}\gamma]\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{5.66}$$

Constitutive theory of orders *m*=1 and *n*=2 for $[{}^{(1)}_{\;\;d}\bar{\sigma}]$

The derivation of the general constitutive theory based on (5.65) requires combined generators $[{}^{\sigma}\underset{\sim}{G}^{i}]$; $i = 1, 2, \ldots, N$ of the symmetric tensors $[{}^{(0)}_{\;\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$, a tensor of rank one, that are of rank two and are also symmetric. Thus, in the current configuration at time $t = t_{n+1}$ we can write

$$[{}^{(1)}_{\;\;d}\bar{\sigma}] = {}^{\sigma}\alpha^{0}[I] + \sum_{i=1}^{N}{}^{\sigma}\alpha^{i}[{}^{\sigma}\underset{\sim}{G}^{i}] \tag{5.67}$$

in which the coefficients ${}^{\sigma}\alpha^{i}$; $i = 0, 1, \ldots, N$ in (5.67) are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \ldots, M$ of $[{}^{(0)}_{\;\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ in the current configuration at time $t = t_{n+1}$. Determination of the coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, N$ in (5.67) using Taylor series expansion of each ${}^{\sigma}\alpha^i$; $i = 0, 1, \dots, N$ follows the standard procedure as used for the general case for the rate constitutive theory of orders (m, n).

Constitutive theory of orders *m*=1 and *n*=2 for ${}^{(0)}\bar{\boldsymbol{q}}$

The constitutive theory for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ based on (5.66) requires the combined generators $\{{}^{q}\underset{\sim}{G}^i\}$; $i = 1, 2, \dots, \widetilde{N}$ of the argument tensors $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$, $[{}^{(2)}\gamma]$ and $\bar{\boldsymbol{g}}$ that are tensors of rank one. Using these we can write the following in the current configuration at time $t = t_{n+1}$.

$$ {}^{(0)}\bar{\boldsymbol{q}} = -\sum_{i=1}^{\widetilde{N}} {}^{q}\alpha^i \{{}^{q}\underset{\sim}{G}^i\} \tag{5.68} $$

in which the coefficients ${}^{q}\alpha^i$; $i = 1, 2, \dots, \widetilde{N}$ are functions of $\bar{\rho}$, $\bar{\theta}$ and the combined invariants ${}^{q\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$ of the argument tensors of ${}^{(0)}\bar{\boldsymbol{q}}$ in the current configuration at time $t = t_{n+1}$. Determination of the coefficients ${}^{q}\alpha^i$; $i = 1, 2, \dots, \widetilde{N}$ using Taylor series expansion about the configuration at time $t = t_n$ follows the procedure described earlier for the general case of the rate constitutive theory of orders (m, n).

5.6.1 Further assumptions and simplifications

We consider (5.65) and (5.66), and eliminate $\bar{\boldsymbol{g}}$ as the argument tensor of $[{}^{(1)}_{\;d}\bar{\sigma}]$. We also eliminate $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$ from the arguments of the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$.

$$ [{}^{(1)}_{\;d}\bar{\sigma}] = [{}^{(1)}_{\;d}\bar{\sigma}(\bar{\rho}\ ,\ [{}^{(0)}_{\;d}\bar{\sigma}]\ ,\ [{}^{(1)}\gamma]\ ,\ [{}^{(2)}\gamma]\ ,\ \bar{\theta})] \tag{5.69} $$

$$ {}^{(0)}\bar{\boldsymbol{q}} = {}^{(0)}\bar{\boldsymbol{q}}(\bar{\rho}\ ,\ \bar{\theta}\ ,\ \bar{\boldsymbol{g}}) \tag{5.70} $$

The constitutive theory for $[{}^{(1)}_{\;d}\bar{\sigma}]$ based on (5.69) requires combined generators of the tensors $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$ that are symmetric tensors of rank two as well as the combined invariants ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \dots, M$ of the same argument tensors $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$.

5.6.2 Oldroyd-B constitutive model for deviatoric Cauchy stress tensor

The simplified rate constitutive theory of order $m = 1$ and $n = 2$ presented in section 5.6.1 can be shown to yield Oldroyd-B constitutive model used for dilute polymeric fluids. The Oldroyd-B constitutive model is a *quasi-linear viscoelastic model*, hence, in order to derive Oldroyd-B constitutive model, we limit the combined generators of the argument tensors of $[{}^{(1)}_{\ d}\bar{\sigma}]$ to $[{}^{(0)}_{\ d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$. The linear combination of the generators resulting from this choice gives us

$$[{}^{(1)}_{\ d}\bar{\sigma}] = {}^{\sigma}\alpha^0[I] + {}^{\sigma}\alpha^1[{}^{(0)}_{\ d}\bar{\sigma}] + {}^{\sigma}\alpha^2[{}^{(1)}\gamma] + {}^{\sigma}\alpha^3[{}^{(2)}\gamma] \tag{5.71}$$

The coefficients ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, 3$ are functions of $\bar{\rho}_{t_{n+1}}$, $\bar{\theta}_{t_{n+1}}$ and the combined invariants of $[{}^{(0)}_{\ d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$ in the current configuration at time $t = t_{n+1}$. We consider Taylor series expansions of ${}^{\sigma}\alpha^i$; $i = 0, 1, \ldots, 3$ about the configuration at time $t = t_n$ in $\bar{\theta}$ and ${}^{\sigma}\underset{\sim}{I}^j$; $j = 1, 2, \ldots, M$ and retain only up to linear terms in $\bar{\theta}$ and the invariants.

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{M} \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\bigg|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ ; \ i = 0, 1, \ldots, 3 \tag{5.72}$$

${}^{\sigma}\alpha^i\big|_{t_n}$, $\frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}\big|_{t_n}$; $j = 1, 2, \ldots, M$ and $\frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\big|_{t_n}$; $i = 0, 1, \ldots, 3$ are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \ldots, M$, however ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big(\bar{\rho}_{t_n}, \bar{\theta}_{t_n}, ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j = 1, 2, \ldots, M, \bar{\theta}_{t_{n+1}}, ({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}; j = 1, 2, \ldots, M\big)$; $i = 0, 1, \ldots, 3$ in (5.72). If we let ${}^{\sigma}\alpha^i{}_{,j} = \frac{\partial({}^{\sigma}\alpha^i)}{\partial({}^{\sigma}\underset{\sim}{I}^j)}$; $j = 1, 2, \ldots, M$, then (5.72) can be written as

$${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i\big|_{t_n} + \sum_{j=1}^{M} ({}^{\sigma}\alpha^i{}_{,j})\big|_{t_n} \big(({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^i)}{\partial\bar{\theta}}\bigg|_{t_n} (\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n}) \ ; \ i = 0, 1, \ldots, 3 \tag{5.73}$$

Substituting from (5.73) in (5.71), we obtain the most general expression for the constitutive theory for $[{}^{(1)}_{\;d}\bar{\sigma}]$ based on the choice of generators in (5.71) and invariants ${}^{\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, M$.

$$
\begin{aligned}
[{}^{(1)}_{\;d}\bar{\sigma}] = &\Big({}^{\sigma}\alpha^{0}\big|_{t_n} + \sum_{j=1}^{M} ({}^{\sigma}\alpha^{0},_{j})\big|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{0})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] + \\
&\Big({}^{\sigma}\alpha^{1}\big|_{t_n} + \sum_{j=1}^{M} ({}^{\sigma}\alpha^{1},_{j})\big|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{1})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(0)}_{\;d}\bar{\sigma}] + \\
&\Big({}^{\sigma}\alpha^{2}\big|_{t_n} + \sum_{j=1}^{M} ({}^{\sigma}\alpha^{2},_{j})\big|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{2})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(1)}\gamma] + \\
&\Big({}^{\sigma}\alpha^{3}\big|_{t_n} + \sum_{j=1}^{M} ({}^{\sigma}\alpha^{3},_{j})\big|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{3})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[{}^{(2)}\gamma]
\end{aligned}
\tag{5.74}
$$

(a) Further assumptions and simplifications

We make the following assumptions to derive the Oldroyd-B model, which is a quasi-linear viscoelastic model:

(i) We delete the terms containing products of the generators $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$ with the invariants ${}^{\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \dots, M$ in the current configuration.

(ii) We also delete the terms containing the products of the generators $[{}^{(0)}_{\;d}\bar{\sigma}]$, $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$ with $(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})$ in the current configuration. This gives

$$
\begin{aligned}
[{}^{(1)}_{\;d}\bar{\sigma}] = &\Big({}^{\sigma}\alpha^{0}\big|_{t_n} + \sum_{j=1}^{M} ({}^{\sigma}\alpha^{0},_{j})\big|_{t_n}\big(({}^{\sigma}\underset{\sim}{I}^{j})_{t_{n+1}} - ({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\big) + \frac{\partial({}^{\sigma}\alpha^{0})}{\partial\bar{\theta}}\bigg|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})\Big)[I] \\
&+\Big({}^{\sigma}\alpha^{1}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{1},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\Big)[{}^{(0)}_{\;d}\bar{\sigma}] \\
&+\Big({}^{\sigma}\alpha^{2}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{2},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\Big)[{}^{(1)}\gamma] \\
&+\Big({}^{\sigma}\alpha^{3}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{3},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}\Big)[{}^{(2)}\gamma]
\end{aligned}
\tag{5.75}
$$

We collect terms and define material coefficients and others. Let

$$
\begin{aligned}
{}^{\sigma}b^{0} = &\ {}^{\sigma}\alpha^{0}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{0},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \quad ; \quad {}^{\sigma}b^{1}_{j} = ({}^{\sigma}\alpha^{0},_{j})\big|_{t_n} \ ; \ j = 1, 2, \dots, M \\
{}^{\sigma}b^{2} = &\ {}^{\sigma}\alpha^{1}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{1},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \quad ; \quad {}^{\sigma}b^{3} = {}^{\sigma}\alpha^{2}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{2},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n} \\
{}^{\sigma}b^{4} = &\ {}^{\sigma}\alpha^{3}\big|_{t_n} - \sum_{j=1}^{M} ({}^{\sigma}\alpha^{3},_{j})\big|_{t_n}({}^{\sigma}\underset{\sim}{I}^{j})_{t_n}
\end{aligned}
\tag{5.76}
$$

Then we can write

$$
\begin{aligned}
[{}^{(1)}_{\;d}\bar{\sigma}] = & {}^{\sigma}b^0[I] + \sum_{j=1}^{M} {}^{\sigma}b^1_j({}^{\sigma}\underset{\sim}{I}^j)_{t_{n+1}}[I] + {}^{\sigma}b^2[{}^{(0)}_{\;d}\bar{\sigma}] + {}^{\sigma}b^3[{}^{(1)}\gamma] \\
& + {}^{\sigma}b^4[{}^{(2)}\gamma] + \left.\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\right|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.77}
$$

We note that the coefficients ${}^{\sigma}b^0$, ${}^{\sigma}b^1_j$; $j = 1, 2, \dots, M$, ${}^{\sigma}b^2$, ${}^{\sigma}b^3$, ${}^{\sigma}b^4$ and $\left.\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\right|_{t_n}$ defined in the configuration at time $t = t_n$ and hence are functions of $\bar{\rho}_{t_n}$, $\bar{\theta}_{t_n}$ and $({}^{\sigma}\underset{\sim}{I}^j)_{t_n}$; $j = 1, 2, \dots, M$.

(iii) (5.77) can be further simplified. In order to derive the Oldroyd-B constitutive model, we retain the invariants $({}^{\sigma}\underset{\sim}{I}^1)_{t_{n+1}} = (i_{{}^{(0)}_{\;d}\bar{\sigma}})_{t_{n+1}} = \text{tr}([{}^{(0)}_{\;d}\bar{\sigma}])$, $({}^{\sigma}\underset{\sim}{I}^2)_{t_{n+1}} = (i_{{}^{(1)}\gamma})_{t_{n+1}} = \text{tr}([{}^{(1)}\gamma])$ and $({}^{\sigma}\underset{\sim}{I}^3)_{t_{n+1}} = (i_{{}^{(2)}\gamma})_{t_{n+1}} = \text{tr}([{}^{(2)}\gamma])$ and discard all others. This reduces (5.77) to the following:

$$
\begin{aligned}
[{}^{(1)}_{\;d}\bar{\sigma}] = & {}^{\sigma}b^0[I] + {}^{\sigma}b^1_1\text{tr}([{}^{(0)}_{\;d}\bar{\sigma}])[I] + {}^{\sigma}b^1_2\text{tr}([{}^{(1)}\gamma])[I] + {}^{\sigma}b^1_3\text{tr}([{}^{(2)}\gamma])[I] + {}^{\sigma}b^2[{}^{(0)}_{\;d}\bar{\sigma}] \\
& + {}^{\sigma}b^3[{}^{(1)}\gamma] + {}^{\sigma}b^4[{}^{(2)}\gamma] + \left.\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\right|_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.78}
$$

which can be written as

$$
\begin{aligned}
[{}^{(0)}_{\;d}\bar{\sigma}] + \Big(-\frac{1}{{}^{\sigma}b^2}\Big)[{}^{(1)}_{\;d}\bar{\sigma}] = & \Big(-\frac{{}^{\sigma}b^0}{{}^{\sigma}b^2}\Big)[I] + \Big(-\frac{{}^{\sigma}b^3}{{}^{\sigma}b^2}\Big)[{}^{(1)}\gamma] + \Big(-\frac{{}^{\sigma}b^4}{{}^{\sigma}b^2}\Big)[{}^{(2)}\gamma] \\
& + \Big(-\frac{{}^{\sigma}b^1_2}{{}^{\sigma}b^2}\Big)\text{tr}([{}^{(1)}\gamma])[I] + \Big(-\frac{{}^{\sigma}b^1_1}{{}^{\sigma}b^2}\Big)\text{tr}([{}^{(0)}_{\;d}\bar{\sigma}])[I] \\
& + \Big(-\frac{{}^{\sigma}b^1_3}{{}^{\sigma}b^2}\Big)\text{tr}([{}^{(2)}\gamma])[I] - \Big(\left.\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\right|_{t_n}\frac{1}{{}^{\sigma}b^2}\Big)(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.79}
$$

We introduce new notations for the material coefficients to conform to the standard notations used in the literature. Let

$$
\begin{aligned}
\Big(-\frac{1}{{}^{\sigma}b^2}\Big) &= (\lambda_1)_{t_n} = \lambda_1\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)\\
\Big(-\frac{{}^{\sigma}b^0}{{}^{\sigma}b^2}\Big) &= \bar{\sigma}_0\big|_{t_n} = \bar{\sigma}_0\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)\\
\Big(-\frac{{}^{\sigma}b^3}{{}^{\sigma}b^2}\Big) &= 2\eta_{t_n} = 2\eta\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)\\
\Big(-\frac{{}^{\sigma}b_2^1}{{}^{\sigma}b^2}\Big) &= \kappa_{t_n} = \kappa\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)\\
\Big(-\frac{{}^{\sigma}b_1^1}{{}^{\sigma}b^2}\Big) &= {}^1\kappa_{t_n} = {}^1\kappa\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)\\
\Big(-\frac{{}^{\sigma}b_3^1}{{}^{\sigma}b^2}\Big) &= {}^2\kappa_{t_n} = {}^2\kappa\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)\\
\Big(\frac{\partial({}^{\sigma}\alpha^0)}{\partial\bar{\theta}}\Big|_{t_n}\frac{1}{{}^{\sigma}b^2}\Big) &= (\alpha_{tm})_{t_n} = \alpha_{tm}\Big(\bar{\rho}_{t_n}\ ,\ \bar{\theta}_{t_n}\ ,\ ({}^{\sigma}\underset{\sim}{I}^j)_{t_n}; j=1,2,\dots,M\Big)
\end{aligned}
\tag{5.80}
$$

then (5.79) can be written as

$$
\begin{aligned}
[{}^{(0)}_{\ d}\bar{\sigma}] + (\lambda_1)_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] =& \bar{\sigma}_0\big|_{t_n}[I] + 2\eta_{t_n}[{}^{(1)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I] + {}^1\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])[I]\\
&+ {}^2\kappa_{t_n}\text{tr}([{}^{(2)}\gamma])[I] + \Big(-\frac{{}^{\sigma}b^4}{{}^{\sigma}b^2}\Big)[{}^{(2)}\gamma] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.81}
$$

We note that $[{}^{(1)}\gamma]$ and $[{}^{(2)}\gamma]$ are first and second convected time derivatives of the strain tensor (in a chosen basis), thus dimensionally (or in terms of units), if we multiply $[{}^{(2)}\gamma]$ by time we obtain units of $[{}^{(1)}\gamma]$. Since $[{}^{(1)}\gamma]$ is already multiplied with $2\eta_{t_n}$, thus $[{}^{(2)}\gamma]$ can be multiplied with $2\eta_{t_n}(\lambda_2)_{t_n}$, where $(\lambda_2)_{t_n}$ is a time constant. Therefore we can choose $(-{}^{\sigma}b^4/{}^{\sigma}b^2) = 2\eta_{t_n}(\lambda_2)_{t_n}$. With this choice (5.81) can be written as

$$
\begin{aligned}
[{}^{(0)}_{\ d}\bar{\sigma}] + (\lambda_1)_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] =& \bar{\sigma}^0\big|_{t_n}[I] + 2\eta_{t_n}[{}^{(1)}\gamma] + 2\eta_{t_n}(\lambda_2)_{t_n}[{}^{(2)}\gamma] + \kappa_{t_n}\text{tr}([{}^{(1)}\gamma])[I]\\
&+ {}^1\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])[I] + {}^2\kappa_{t_n}\text{tr}([{}^{(2)}\gamma])[I] - (\alpha_{tm})_{t_n}(\bar{\theta}_{t_{n+1}} - \bar{\theta}_{t_n})[I]
\end{aligned}
\tag{5.82}
$$

(iv) We further assume that: (a) the initial stress field associated with the configuration at time $t = t_n$, i.e., $\bar{\sigma}_0\big|_{t_n}[I]$, is zero (b) the stress field due to thermal expansion and contraction between the current configuration at time $t = t_{n+1}$ and the configuration at time $t = t_n$, i.e., the last term in (5.82), is neglected and (c) we further assume that ${}^1\kappa_{t_n}\text{tr}([{}^{(0)}_{\ d}\bar{\sigma}])$ and

${}^2\kappa_{t_n}\text{tr}([^{(2)}\gamma])$ can be neglected. Thus we obtain the following from (5.82)

$$[^{(0)}_{\ d}\bar{\sigma}] + (\lambda_1)_{t_n}[^{(1)}_{\ d}\bar{\sigma}] = 2\eta_{t_n}[^{(1)}\gamma] + 2\eta_{t_n}(\lambda_2)_{t_n}[^{(2)}\gamma] + \kappa_{t_n}\text{tr}([^{(1)}\gamma])[I] \tag{5.83}$$

$(\lambda_1)_{t_n}$ is called *relaxation time*, η_{t_n} is *first viscosity*, κ_{t_n} is *second viscosity*, $(\lambda_2)_{t_n}$ is *retardation time* and $(\alpha_{tm})_{t_n}$ is *thermal modulus*. (5.83) is the *Oldroyd-B constitutive model for compressible thermoviscoelastic fluids* in which $(\lambda_1)_{t_n}$, η_{t_n}, κ_{t_n}, $(\lambda_2)_{t_n}$ and $(\alpha_{tm})_{t_n}$ can be deformation dependent during evolutions.

(b) Remarks:

1. The coefficients $(\lambda_1)_{t_n}$, $(\lambda_2)_{t_n}$, η_{t_n}, κ_{t_n}, $(\alpha_{tm})_{t_n}$ etc. are defined in the configuration at time $t = t_n$ for which deformation is known, whereas all other quantities in (5.82) and (5.83) are in the current configuration at time $t = t_{n+1}$. This is a consequence of Taylor series expansion of the coefficients about the configuration at time $t = t_n$.

2. Based on (5.80), $(\lambda_1)_{t_n}$, $(\lambda_2)_{t_n}$, η_{t_n}, κ_{t_n} etc. can be deformation dependent during evolution (keeping 1. in mind) permitting experimental and/or empirical description of $(\lambda_1)_{t_n}$, $(\lambda_2)_{t_n}$, η_{t_n}, κ_{t_n} etc. using their arguments given in (5.80). Thus, power law, Sutherland law etc. for $(\lambda_1)_{t_n}$, $(\lambda_2)_{t_n}$, η_{t_n}, κ_{t_n} etc. dependent on $\bar{\rho}_{t_n}$ and $\bar{\theta}_{t_n}$ are valid. Dependence of $(\lambda_1)_{t_n}$, $(\lambda_2)_{t_n}$, η_{t_n}, κ_{t_n} etc. on the combined invariants of $[^{(0)}_{\ d}\bar{\sigma}]$, $[^{(1)}\gamma]$ and $[^{(2)}\gamma]$ allows us to represent variable shear thinning and shear thickening behaviors during the evolution. Power law, Carreau-Yasuda models etc. are valid based on (5.80) (contrary to the belief that these models do not have continuum mechanics foundation [22]).

3. By replacing ($[^{(1)}_{\ d}\bar{\sigma}]$, $[^{(0)}_{\ d}\bar{\sigma}]$, $[^{(1)}\gamma]$, $[^{(2)}\gamma]$) with ($[_d\bar{\sigma}^{(1)}]$, $[_d\bar{\sigma}^{(0)}]$, $[\gamma^{(1)}]$, $[\gamma^{(2)}]$), ($[_d\bar{\sigma}_{(1)}]$, $[_d\bar{\sigma}_{(0)}]$, $[\gamma_{(1)}]$, $[\gamma_{(2)}]$) and ($[^{(1)}_{\ d}\bar{\sigma}^J]$, $[^{(0)}_{\ d}\bar{\sigma}^J]$, $[^{(1)}\gamma^J]$, $[^{(2)}\gamma^J]$) in (5.82) and (5.83) (including coefficients), we obtain the Oldroyd-B model that correspond to contravariant basis (*upper convected*), covariant basis (*lower convected*) and Jaumann rates. It is rather obvious that when the

deformation is finite, all three rate constitutive equations will represent different physics, upper convected case being in most agreement with the physics of finite deformation [30].

4. If we assume that the configuration at times t_n and t_{n+1} are in close proximity of each other, then in (5.80) all material coefficients can be expressed in terms of their arguments at time $t = t_{n+1}$ instead of at time $t = t_n$ (as shown in (5.80)). This is what is done in currently used models for variable material coefficients [22, 38, 39]. This is obviously not supported by the derivation presented here. When t_n in (5.80) is replaced by t_{n+1}, the material coefficients become functions of the unknown deformation field in the current configuration.

5. The constitutive theories for the heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ have already been presented. Generally, Fourier heat conduction law (section 5.5.1) is commonly used in the majority of the published work.

5.7 Rate constitutive theories for incompressible thermoviscoelastic fluids i.e. polymeric liquids

In all derivations of the rate constitutive theory presented so far for thermoviscoelastic fluids of orders (m, n), $(1, 1)$ and $(1, 2)$ we have considered the fluid to be compressible. In this section we consider incompressible thermoviscoelastic fluids. For incompressible matter

$$\bar{\rho} = \rho_0 = \text{constant} \tag{5.84}$$

$$\text{div}(\bar{\boldsymbol{v}}) = 0 \tag{5.85}$$

$$\therefore \quad \text{tr}([{}^{(1)}\gamma]) = \text{tr}([\gamma^{(1)}]) = \text{tr}([\gamma_{(1)}]) = \text{tr}([\gamma^J]) = \text{tr}([\bar{D}]) = 0 \tag{5.86}$$

$$\det([J]) = 1 \tag{5.87}$$

As in this case $\bar{\rho} = \rho_0$, density $\bar{\rho}$ can be eliminated from the argument tensors of the dependent variables $[{}^{(1)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ in the rate constitutive theory for incompressible thermoviscoelastic fluids.

The rate theories of orders (m,n), $(1,1)$ and $(1,2)$ and their simplifications can be easily modified to hold for incompressible case by: (i) eliminating $\bar{\rho}$ all together from the entire derivations (ii) by incorporating incompressibility condition ((5.84) - (5.87) as appropriate). Details are straight forward and hence not presented here for the sake of brevity.

The simplifications of the rate constitutive theories of order $(1,1)$ and $(1,2)$ for compressible thermoviscous fluids leading to Maxwell model, Giesekus model and Oldroyd-B model for compressible thermoviscous fluids can also be modified accordingly by simply using (5.86) in (5.48), (5.63) and (5.83).

Maxwell model:

$$[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] = 2\eta_{t_n}[{}^{(1)}\gamma] \tag{5.88}$$

Giesekus model:

$$[{}^{(0)}_{\ d}\bar{\sigma}] + \lambda_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] = 2\eta_{t_n}[{}^{(1)}\gamma] + \frac{\lambda_{t_n}}{\eta_{t_n}}\alpha[{}^{(0)}_{\ d}\bar{\sigma}]^2 \tag{5.89}$$

Oldroyd-B model:

$$[{}^{(0)}_{\ d}\bar{\sigma}] + (\lambda_1)_{t_n}[{}^{(1)}_{\ d}\bar{\sigma}] = 2\eta_{t_n}[{}^{(1)}\gamma] + 2\eta_{t_n}(\lambda_2)_{t_n}[{}^{(2)}\gamma] \tag{5.90}$$

The remarks presented for these models for compressible case regarding variable coefficients (and others) hold for (5.88) - (5.90) as well except that the fact that $\bar{\rho}$ is eliminated all together and the incompressibility condition (5.86) is enforced. By choosing compatible stress and strain rate tensors in the basis of choice in (5.88) - (5.90), it is straight forward to obtain their forms in contravariant basis (upper convected), covariant basis (lower convected) and those using Jaumann rates. The Fourier heat conduction law is commonly used as a constitutive theory for heat vector. However, the other rate theories of any desired order (m,n) for the heat vector can be easily derived using the derivation presented for compressible case by eliminating $\bar{\rho}$ and by imposing incompressibility constraint (5.86).

5.8 Numerical studies using Giesekus constitutive model

In this section we consider fully developed flow of a incompressible Giesekus fluid between parallel plates as model problem. We use contravariant Cauchy stress tensor and Almansi strain tensors as conjugate measures of the stress and strain tensors in Eulerian description. If we decompose the contravariant Cauchy stress tensor in equilibrium stress and deviatoric contravariant Cauchy stress tensor, then the equilibrium stress is mechanical pressure p and the deviatoric contravariant Cauchy stress tensor becomes a dependent variable in the constitutive theory. This yields upper convected Giesekus (UCG1) constitutive model. Numerical results are presented using the upper convected Giesekus constitutive model derived in previous section (UCG1) as well as using the currently used constitutive model in deviatoric polymer stress (UCG2).

Since the description is understood to be Eulerian, we drop over bar $(^{-})$ on all quantities for simplicity of notation and replace it with hat $(\hat{})$ to emphasize that these quantities have dimensions. Quantities without hat $(\hat{})$ are dimensionless. To conform to commonly used engineering notations we replace x_i ; $i = 1,2,3$ by x,y,z and $\bar{v}_i$; $i = 1,2,3$ by u,v,w and ${}^{(0)}_{\;d}\bar{\sigma}_{ij}$; $i,j = 1,2,3$ by τ_{ij} ; $i,j = 1,2,3$ in the mathematical model. We consider an incompressible Giesekus fluid PIB/C14 [65] with the following material coefficients (assumed constant).

$$\hat{\rho} = 800 \text{ kg/m}^3 \quad ; \quad \hat{\eta}_s = 0.002 \text{ Pa s} \quad ; \quad \hat{\eta}_p = 1.424 \text{ Pa s}$$
$$\hat{\eta} = 1.426 \text{ Pa s} \quad ; \quad \hat{\lambda} = 0.06 \text{ s} \quad ; \quad \alpha = 0.15$$

in which $\hat{\rho}$, $\hat{\eta}_s$, $\hat{\eta}_p$, $\hat{\eta}$, $\hat{\lambda}$ and α are density, solvent viscosity, polymer viscosity, total viscosity, relaxation time and mobility factor.

For a fixed configuration and a given fluid we can study the influence of the constitutive models on the flow physics in at least two ways:

(i) For a fixed flow rate, the differences in the constitutive equation in the two models will produce different $\partial p/\partial x$ and other dependent variables in the two cases. As the flow rate increases, the differences in $\partial p/\partial x$ and the other dependent variables in the two cases are expected to increase as well.

(ii) In the second approach, we could choose a value of $\partial p/\partial x$ that is the same in both cases and compute results. Both models are bound to produce different velocity fields and hence different flow rates. For very low values of $\partial p/\partial x$ we expect the velocity field in the two cases to be not drastically different from each other but as $\partial p/\partial x$ increases, the differences are expected to be significant.

Obviously, (ii) is easier as it merely requires specification of $\partial p/\partial x$ as input and the rest of the detail of the flow are computed. We use this approach to study the influence of the two constitutive models (UCG1 and UCG2) on the flow physics of fully developed flow between parallel plates (model problem 1) and fully developed flow between parallel plates using a two dimensional formulation (model problem 2). It is obvious that both model problems will be in agreement when the same constitutive model is used.

5.8.1 Model Problem 1: fully developed flow between parallel plates

Figure 5.1 shows a schematic using dimensionless quantities. The plates are separated by a distance $2H$. The origin of the xy-coordinate is located at the center of the plates and the positive x-direction is the direction of the flow. The flow is pressure driven, i.e. $\partial p/\partial x$ (negative) is specified. The mathematical model describing the flow physics (for incompressible case with isothermal flow assumption) consists of x- and y-momentum equations and the constitutive equations. The continuity equation in this case is satisfied identically.

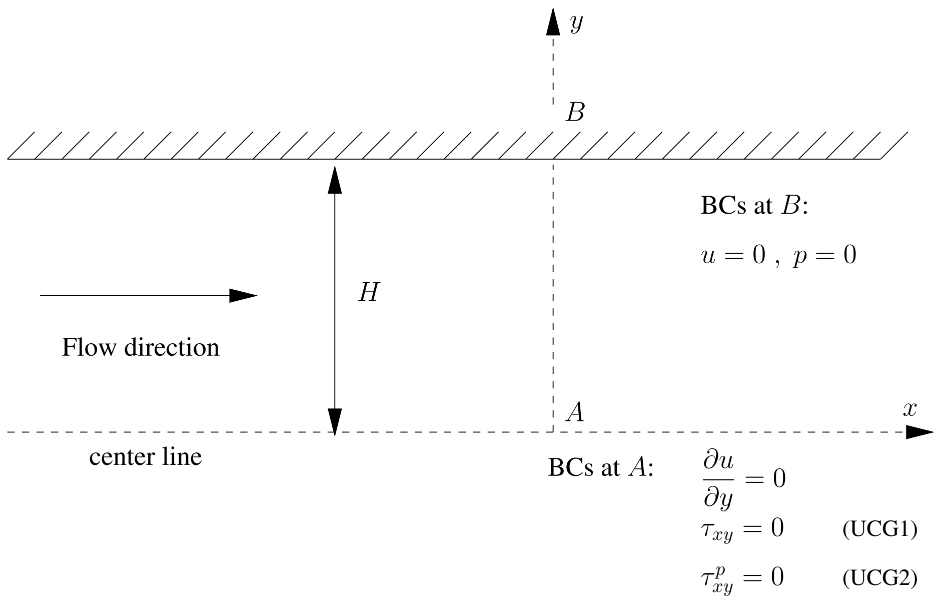


Figure 5.1: Schematic of 1-D fully developed flow between parallel plates (half domain)

We begin with all quantities with their usual dimensions (units) in the development of the mathematical model and then non-dimensionalize them using the following. The quantities with the subscript zero are the reference quantities.

$$
\begin{aligned}
& x=\frac{\hat{x}}{L_0}\ ,\ y=\frac{\hat{y}}{L_0}\ ,\ \eta=\frac{\hat{\eta}}{\eta_0}\ ,\ \eta_s=\frac{\hat{\eta}_s}{\eta_0}\ ,\ \eta_p=\frac{\hat{\eta}_p}{\eta_0}\ ,\ \rho=\frac{\hat{\rho}}{\rho_0}\ ,\ u=\frac{\hat{u}}{u_0} \\
& v=\frac{\hat{v}}{u_0}\ ,\ p=\frac{\hat{p}}{p_0}\ ,\ \boldsymbol{\tau}=\frac{\hat{\boldsymbol{\tau}}}{\tau_0}\ ,\ p_0=\tau_0=\begin{cases} \rho_0 u_0^2 & ;\ \text{Ch. kinetic energy} \\ & \text{or} \\ \dfrac{\mu_0 u_0}{L_0} & ;\ \text{Ch. viscous stress} \end{cases}
\end{aligned}
\tag{5.91}
$$

in which $\hat{u}$, $\hat{v}$ are velocities in the x- and y-direction, $\hat{p}$ is mechanical pressure and $\hat{\boldsymbol{\tau}}$ is deviatoric stress tensor, all in the current configuration. We choose the larger of the two for p_0 (and τ_0). This results in dimensionless form of the mathematical model given in the following:

Momentum equations:

In the absence of body forces

$$\left(\frac{p_0}{\rho_0 u_0^2}\right)\frac{\partial p}{\partial x} - \left(\frac{\tau_0}{\rho_0 u_0^2}\right)\frac{\partial \tau_{xy}}{\partial y} = 0 \tag{5.92}$$

$$\left(\frac{p_0}{\rho_0 u_0^2}\right)\frac{\partial p}{\partial y} - \left(\frac{\tau_0}{\rho_0 u_0^2}\right)\frac{\partial \tau_{yy}}{\partial y} = 0 \tag{5.93}$$

Giesekus constitutive model:

We consider the upper convected Giesekus constitutive model derived in section 5.5.3 (UCG1) and the upper convected Giesekus constitutive model used currently (UCG2) [22].

UCG1:

In this model, the first convected time derivative of $\boldsymbol{\tau}$, deviatoric contravariant Cauchy stress tensor, is a dependent variable in the constitutive theory and the dimensionless form is given by

$$\begin{aligned}
&\tau_{xx} - 2De\tau_{xy}\frac{\partial u}{\partial y} - \alpha\frac{De}{\eta}\left(\frac{L_0\tau_0}{u_0\eta_0}\right)\left((\tau_{xx})^2 + (\tau_{xy})^2\right) = 0 \\
&\tau_{yy} - \alpha\frac{De}{\eta}\left(\frac{L_0\tau_0}{u_0\eta_0}\right)\left((\tau_{yy})^2 + (\tau_{xy})^2\right) = 0 \\
&\tau_{xy} - De\tau_{yy}\frac{\partial u}{\partial y} - \alpha\frac{De}{\eta}\left(\frac{L_0\tau_0}{u_0\eta_0}\right)\tau_{xy}\left(\tau_{xx} + \tau_{yy}\right) = \eta\left(\frac{u_0\eta_0}{L_0\tau_0}\right)\frac{\partial u}{\partial y}
\end{aligned} \tag{5.94}$$

Equations (5.92) - (5.94) constitute the complete mathematical model in dependent variables u, p, τ_{xx}, τ_{yy} and τ_{xy} for fully developed flow between parallel plates when using UCG1.

UCG2:

In this model $\boldsymbol{\tau}$ is decomposed into solvent and polymer stresses.

$$\boldsymbol{\tau} = \boldsymbol{\tau}^s + \boldsymbol{\tau}^p \tag{5.95}$$

The Newton's law of viscosity is assumed as a constitutive model for $\boldsymbol{\tau}^s$. τ^s_{xx} and τ^s_{yy} are zero for this model problem and we only have τ^s_{xy} in the constitute model for solvent stress.

$$\tau^s_{xy} = \Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\frac{\partial u}{\partial y} \tag{5.96}$$

and hence, from (5.95)

$$\tau_{xx} = \tau^p_{xx} \qquad ; \qquad \tau_{yy} = \tau^p_{yy} \qquad ; \qquad \tau_{xy} = \tau^p_{xy} + \Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\frac{\partial u}{\partial y} \tag{5.97}$$

For polymer stress $\boldsymbol{\tau}^p$, the dimensionless form of the constitutive equations are given by (obtained by replacing $\boldsymbol{\tau}$ with $\boldsymbol{\tau}^p$ and η by η_p in (5.94)) the following [22]:

$$\begin{aligned}
&\tau^p_{xx} - 2De\tau^p_{xy}\frac{\partial u}{\partial y} - \alpha\frac{De}{\eta_p}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\big((\tau^p_{xx})^2 + (\tau^p_{xy})^2\big) = 0\\
&\tau^p_{yy} - \alpha\frac{De}{\eta_p}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\big((\tau^p_{yy})^2 + (\tau^p_{xy})^2\big) = 0\\
&\tau^p_{xy} - De\tau^p_{yy}\frac{\partial u}{\partial y} - \alpha\frac{De}{\eta_p}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\tau^p_{xy}\big(\tau^p_{xx} + \tau^p_{yy}\big) = \eta_p\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\frac{\partial u}{\partial y}
\end{aligned} \tag{5.98}$$

Using (5.97) in the momentum equations (5.92) and (5.93), we can express the momentum equations in terms of τ^p_{yy}, τ^p_{xy} and velocity gradients

$$\Big(\frac{p_0}{\rho_0 u_0^2}\Big)\frac{\partial p}{\partial x} - \Big(\frac{\tau_0}{\rho_0 u_0^2}\Big)\frac{\partial \tau^p_{xy}}{\partial y} - \Big(\frac{\eta_0}{L_0\rho_0 u_0}\Big)\eta_s\frac{\partial^2 u}{\partial y^2} = 0 \tag{5.99}$$

$$\Big(\frac{p_0}{\rho_0 u_0^2}\Big)\frac{\partial p}{\partial y} - \Big(\frac{\tau_0}{\rho_0 u_0^2}\Big)\frac{\partial \tau^p_{yy}}{\partial y} = 0 \tag{5.100}$$

(5.98) - (5.100) constitute the complete mathematical model in dependent variables u, p, τ^p_{xx}, τ^p_{yy} and τ^p_{xy} for fully developed flow between parallel plates when using UCG2.

Solutions of the BVPs:

In this section we consider solutions of the BVPs described by (5.92) - (5.94) for UCG1 and (5.98) - (5.100) for UCG2. Since $\partial p/\partial x$ is constant (specified), from (5.92) we can determine τ_{xy} by integrating with respect to y and using the boundary condition $\tau_{xy} = 0$ at $y = 0$ (due to symmetry)

$$\tau_{xy} = \left(\frac{\partial p}{\partial x}\right) y \tag{5.101}$$

A theoretical solution for the remaining dependent variables is not readily possible due to the complexity of the constitutive equations in both boundary value problems (UCG1 and UCG2), hence we consider their numerical solutions using finite element processes based on the residual functional (least squares finite element method). The local approximations are considered in higher order spaces $H^{k,p}(\bar{\Omega}_x^e)$ in which $\bar{\Omega}_x^e$ is the spatial domain of a typical element 'e' of the discretization. The resulting non-linear algebraic equations from the least squares process are solved using Newton's linear method. The computational processes in this approach are unconditionally stable and permit higher order global differentiability local approximations. See reference [61–63] for details of local approximations and the least squares process for non-linear PDEs and higher order spaces. In the computations of the numerical solutions we choose

$$\hat{H} = L_0 = 3.175 \text{ mm} \ ; \ \rho_0 = \hat{\rho} = 800 \text{ kg/m}^3 \ ; \ \eta_0 = \hat{\eta} = 1.426 \text{ Pa s} \ ; \ u_0 = 0.5 \text{ m/s}$$

which gives

$$H = 1 \qquad ; \qquad p_0 = \tau_0 = \rho_0 u_0^2 = 200 \text{ Pa} \qquad ; \qquad Re = \frac{\rho_0 L_0 u_0}{\eta_0} = 0.8906$$

$$De = \frac{\hat{\lambda} u_0}{L_0} = 9.45 \qquad \text{or} \qquad De = \frac{\hat{\lambda} u_{max}}{L_0} = 18.89764 u_{max}$$

A good discretization of the spatial domain $0 \leq y \leq 1$ is important in ensuring satisfactory convergence of the Newton's linear method for the system of non-linear algebraic equations and good accuracy of computed solutions. With progressively increasing $\partial p/\partial x$, we expect development of a constant velocity core at the center of the flow. This suggests the use of a highly biased finer discretization towards the walls. A two element graded mesh with element length of 0.2 and 0.8 starting from the wall (see figure 5.2) is used in and local approximations are p-version (3-node elements) in higher order spaces.

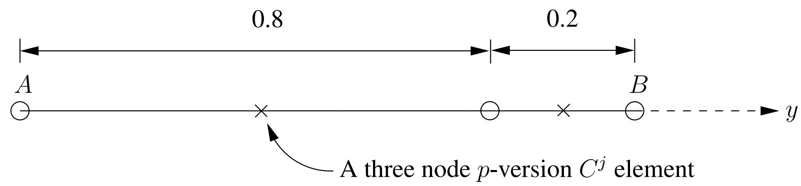


Figure 5.2: Graded mesh discretization using two 3-node p-version elements

Initial p-convergence studies with this discretization suggest $p = 9$ with $k = 2$, local approximations of class $C^1(\bar{\Omega}^e_x)$, to be sufficient for good accuracy of results. For this choice of mesh, p-level ($p = 9$) and order of the space ($k = 2$), the residual or least squares functional values remain $O(10^{-8})$ - $O(10^{-20})$ indicating that the PDEs are satisfied very accurately (in the pointwise sense for UCG1 as the integrals are Riemann, and not strictly in the pointwise sense for UCG2 since the integrals are Lebesgue) when local approximations for u, p, τ_{xx}, τ_{yy} and τ_{xy} are of class $C^1(\bar{\Omega}^e_x)$. Newton's linear method used for solving the non-linear algebraic equations converges in less than 10 iterations for all numerical studies presented here. In the numerical studies we begin with $\partial p/\partial x = -0.1$ for which a converged solution is obtained and then progressively increase it up to $\partial p/\partial x = -0.275$ using a continuation procedure in which converged solutions at lower $\partial p/\partial x$ are used as initial (or starting) solution in the Newton's linear method. Figure 5.3 shows graphs of velocity u versus y for different values of $\partial p/\partial x$ for both UCG1 and UCG2. Graphs of velocity gradient $\partial u/\partial y$ versus y for different values of $\partial p/\partial x$ are shown in figure 5.4. For $\partial p/\partial x$ values up to -0.2, the results from both UCG1 and UCG2 are in good agreement (figures 5.3 and 5.4). Beyond $\partial p/\partial x$ values of -0.2, the results from the two BVPs begin to deviate. Higher values of $\partial p/\partial x$ result in larger deviations between the two models. At $\partial p/\partial x = -0.275$, u_{max} at $y = 0$ from UCG1 is more than twice of u_{max} at $y = 0$ from UCG2. This of course implies drastically different flow rates resulting from the two models for the same pressure gradient.

Figures 5.5 - 5.7 show plots of τ_{xx}, τ_{yy} and τ_{xy} versus y for both UCG1 and UCG2. For $\partial p/\partial x$

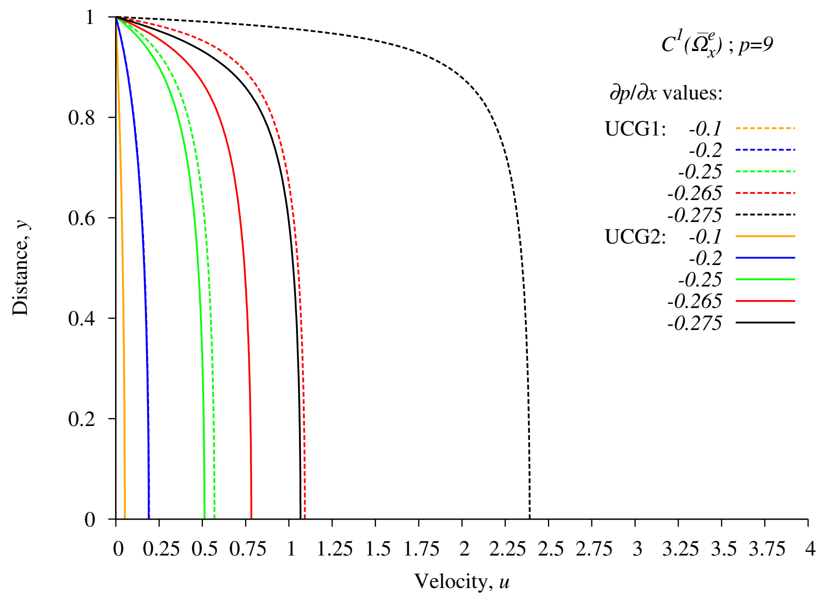


Figure 5.3: Velocity u versus distance y

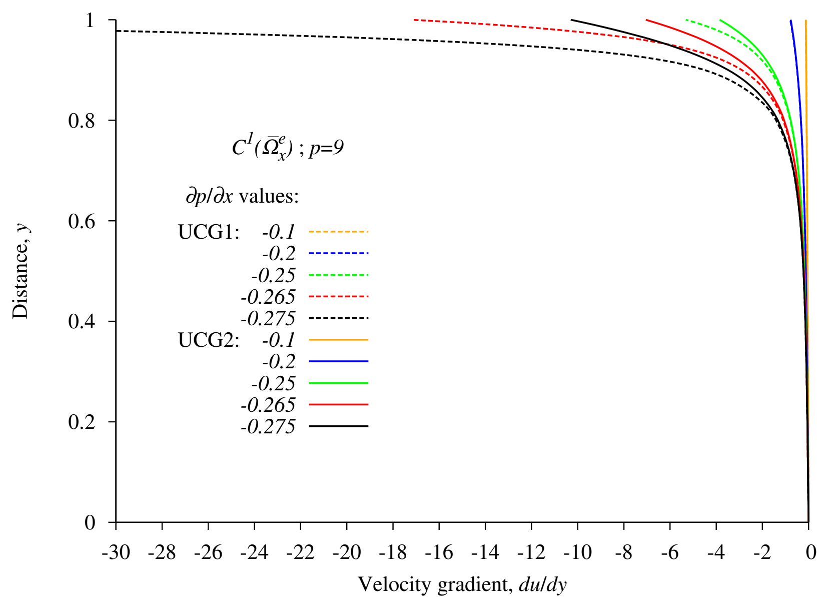


Figure 5.4: Velocity gradient du/dy versus distance y

values beyond -0.2 we observe progressively increasing deviations between solutions obtained from the two BVPs for τ_{xx} and τ_{yy}. For $\partial p/\partial x = -0.275$, τ_{xx} and τ_{yy} from UCG1 are roughly more than twice of those from UCG2. Computed τ_{xy} from the numerical solutions of both BVPs are in perfect agreement with the theoretical solution (5.101) for all values of $\partial p/\partial x$ as τ_{xy} only depends upon $\partial p/\partial x$ which is same in both models. The residual (I) values of $O(10^{-8})$ or lower and the use of $C^1(\bar{\Omega}_x^e)$ ensure that the computed solutions satisfy the GDEs accurately.

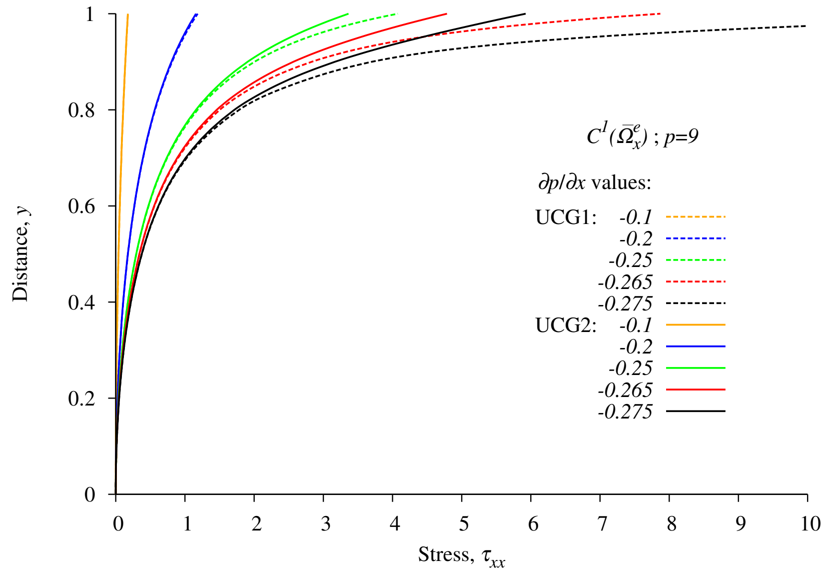


Figure 5.5: Stress component τ_{xx} versus distance y

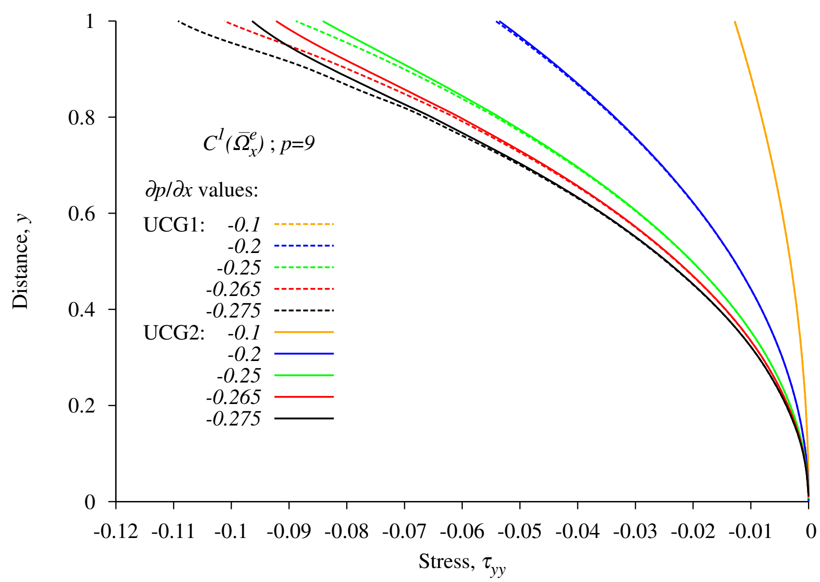


Figure 5.6: Stress component τ_{yy} versus distance y

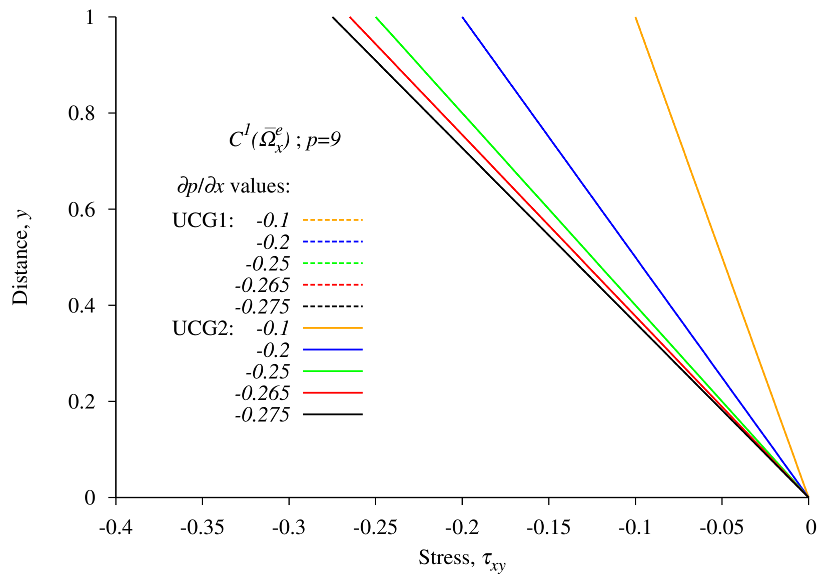


Figure 5.7: Stress component τ_{xy} versus distance y

5.8.2 Model Problem 2: fully developed flow between parallel plates using 2D formulation

We consider the same model problem as considered for model problem 1, i.e. fully developed flow between parallel plates but we use 2D mathematical model. The purpose is to show performance of full mathematical model and to demonstrate that for fully developed flow between parallel plates, this full model produces precisely same results as degenerate model used in section 5.8.1. Figure 5.8 shows a schematic using dimensionless quantities in which $ABCD$ is the computational domain. Origin of the coordinate system x, y is located at A. Positive x-direction is the direction of the flow. In this case, the mathematical model describing the flow physics (for incompressible case with isothermal flow assumption) consists of continuity equation, x- and y-momentum equations and the constitutive equations. We begin with all quantities with their usual dimensions (units) in the development of the mathematical model and then non-dimensionalize them using (5.91) in which $\hat{u}$, $\hat{v}$ are velocities in the x- and y-direction, $\hat{p}$ is mechanical pressure and $\hat{\boldsymbol{\tau}}$ is deviatoric stress tensor, all in the current configuration. We choose the larger of the two for p_0 (and τ_0). This results in the following dimensionless form of the mathematical model:

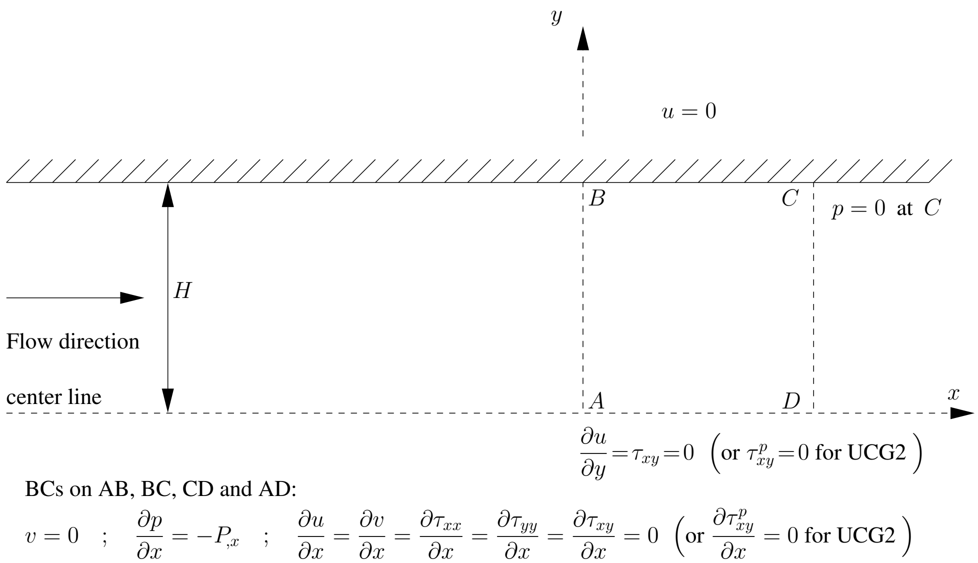


Figure 5.8: Schematic of 2-D fully developed flow between parallel plates (half domain)

Continuity equation:

$$\rho\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) = 0 \tag{5.102}$$

Momentum equations:

In the absence of body forces

$$\begin{aligned}
\rho\Big(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) + \Big(\frac{p_0}{\rho_0 u_0^2}\Big)\frac{\partial p}{\partial x} - \Big(\frac{\tau_0}{\rho_0 u_0^2}\Big)\Big(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}\Big) = 0 \\
\rho\Big(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) + \Big(\frac{p_0}{\rho_0 u_0^2}\Big)\frac{\partial p}{\partial y} - \Big(\frac{\tau_0}{\rho_0 u_0^2}\Big)\Big(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\Big) = 0
\end{aligned} \tag{5.103}$$

Giesekus constitutive model:

We consider the upper convected Giesekus constitutive model derived in section 5.5.3 (UCG1) and the upper convected Giesekus constitutive model used currently (UCG2).

UCG1:

In this model, the first convected time derivative of $\boldsymbol{\tau}$, the deviatoric contravariant Cauchy stress tensor, is a dependent variable in the constitutive theory and dimensionless form is given by

$$
\begin{aligned}
&\tau_{xx}+De\Big(u\frac{\partial\tau_{xx}}{\partial x}+v\frac{\partial\tau_{xx}}{\partial y}-2\tau_{xy}\frac{\partial u}{\partial y}-2\tau_{xx}\frac{\partial u}{\partial x}\Big)\\
&\qquad-\alpha\frac{De}{\eta}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\big((\tau_{xx})^2+(\tau_{xy})^2\big)=2\eta\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\frac{\partial u}{\partial x}\\
&\tau_{yy}+De\Big(u\frac{\partial\tau_{yy}}{\partial x}+v\frac{\partial\tau_{yy}}{\partial y}-2\tau_{xy}\frac{\partial v}{\partial x}-2\tau_{yy}\frac{\partial v}{\partial y}\Big)\\
&\qquad-\alpha\frac{De}{\eta}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\big((\tau_{yy})^2+(\tau_{xy})^2\big)=2\eta\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\frac{\partial v}{\partial y}\\
&\tau_{xy}+De\Big(u\frac{\partial\tau_{xy}}{\partial x}+v\frac{\partial\tau_{xy}}{\partial y}-\tau_{xy}\big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\big)-\tau_{xx}\frac{\partial v}{\partial x}-\tau_{yy}\frac{\partial u}{\partial y}\Big)\\
&\qquad-\alpha\frac{De}{\eta}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\tau_{xy}\big(\tau_{xx}+\tau_{yy}\big)=\eta\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)
\end{aligned}
\tag{5.104}
$$

Equations (5.102) - (5.104) constitute the complete mathematical model in dependent variables u, v, p, τ_{xx}, τ_{yy} and τ_{xy} for two dimensional steady flow using the constitutive model UCG1.

UCG2:

This constitutive model is used currently [22]. As in model problem 1, here also, $\boldsymbol{\tau}$ is decomposed into solvent and polymer stresses.

$$
\boldsymbol{\tau}=\boldsymbol{\tau}^s+\boldsymbol{\tau}^p \tag{5.105}
$$

and the Newton's law of viscosity is assumed as a constitutive theory for $\boldsymbol{\tau}^s$.

$$
\tau^s_{xx}=2\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\frac{\partial u}{\partial x}\ ;\ \tau^s_{yy}=2\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\frac{\partial v}{\partial y}\ ;\ \tau^s_{xy}=\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big) \tag{5.106}
$$

and hence, from (5.105)

$$\begin{aligned}&\tau_{xx}=\tau^p_{xx}+2\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\frac{\partial u}{\partial x} \quad ; \quad \tau_{yy}=\tau^p_{yy}+2\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\frac{\partial v}{\partial y}\\&\tau_{xy}=\tau^p_{xy}+\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\eta_s\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\end{aligned} \tag{5.107}$$

For polymer stress $\boldsymbol{\tau}^p$, the dimensionless form of the constitutive equations are given by (obtained by replacing $\boldsymbol{\tau}$ with $\boldsymbol{\tau}^p$ and η by η_p in (5.104)) the following [22]:

$$\begin{aligned}&\tau^p_{xx}+De\Big(u\frac{\partial\tau^p_{xx}}{\partial x}+v\frac{\partial\tau^p_{xx}}{\partial y}-2\tau^p_{xy}\frac{\partial u}{\partial y}-2\tau^p_{xx}\frac{\partial u}{\partial x}\Big)\\&\qquad-\alpha\frac{De}{\eta_p}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\big((\tau^p_{xx})^2+(\tau^p_{xy})^2\big)=2\eta_p\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\frac{\partial u}{\partial x}\\&\tau^p_{yy}+De\Big(u\frac{\partial\tau^p_{yy}}{\partial x}+v\frac{\partial\tau^p_{yy}}{\partial y}-2\tau^p_{xy}\frac{\partial v}{\partial x}-2\tau^p_{yy}\frac{\partial v}{\partial y}\Big)\\&\qquad-\alpha\frac{De}{\eta_p}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\big((\tau^p_{yy})^2+(\tau^p_{xy})^2\big)=2\eta_p\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\frac{\partial v}{\partial y}\\&\tau^p_{xy}+De\Big(u\frac{\partial\tau^p_{xy}}{\partial x}+v\frac{\partial\tau^p_{xy}}{\partial y}-\tau^p_{xy}\big(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\big)-\tau^p_{xx}\frac{\partial v}{\partial x}-\tau^p_{yy}\frac{\partial u}{\partial y}\Big)\\&\qquad-\alpha\frac{De}{\eta_p}\Big(\frac{L_0\tau_0}{u_0\eta_0}\Big)\tau^p_{xy}\big(\tau^p_{xx}+\tau^p_{yy}\big)=\eta_p\Big(\frac{u_0\eta_0}{L_0\tau_0}\Big)\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)\end{aligned} \tag{5.108}$$

Using (5.107) in the momentum equations (5.103), we can express the momentum equations in terms of τ^p_{xx}, τ^p_{yy}, τ^p_{xy} and velocity gradients

$$\begin{aligned}&\rho\Big(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\Big)+\Big(\frac{p_0}{\rho_0u_0^2}\Big)\frac{\partial p}{\partial x}-\Big(\frac{\tau_0}{\rho_0u_0^2}\Big)\Big(\frac{\partial\tau^p_{xx}}{\partial x}+\frac{\partial\tau^p_{xy}}{\partial y}\Big)\\&\qquad\qquad-\Big(\frac{\eta_0}{L_0\rho_0u_0}\Big)\eta_s\Big(2\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2v}{\partial y\partial x}\Big)=0\\&\rho\Big(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\Big)+\Big(\frac{p_0}{\rho_0u_0^2}\Big)\frac{\partial p}{\partial y}-\Big(\frac{\tau_0}{\rho_0u_0^2}\Big)\Big(\frac{\partial\tau^p_{xy}}{\partial x}+\frac{\partial\tau^p_{yy}}{\partial y}\Big)\\&\qquad\qquad-\Big(\frac{\eta_0}{L_0\rho_0u_0}\Big)\eta_s\Big(2\frac{\partial^2v}{\partial y^2}+\frac{\partial^2v}{\partial x^2}+\frac{\partial^2u}{\partial y\partial x}\Big)=0\end{aligned} \tag{5.109}$$

Equations (5.102), (5.108), (5.109) constitute the complete mathematical model in dependent variables u, v, p, τ^p_{xx}, τ^p_{yy} and τ^p_{xy} for two dimensional steady flow using the constitutive model UCG2, used currently for incompressible Giesekus fluids.

Solutions of the BVPs:

In this section we consider solutions of the BVPs described by(5.102) - (5.104) for UCG1 and (5.102), (5.108), (5.109) for UCG2. A theoretical solution for dependent variables is not possible due to the complexity of the constitutive equations in both boundary value problems, hence we consider their numerical solutions using finite element processes based on the residual functional (least squares finite element method) in which the resulting non-linear algebraic equations from the least squares process are solved using Newton's linear method. The computational processes in this approach are unconditionally stable and permit higher order global differentiability local approximations. Details of the local approximations and the least squares finite element processes for non-linear PDEs and higher order spaces can be found in references [61–63, 66–68]. The local approximations are considered in higher order spaces $H^{k,p}(\bar{\Omega}^e_{xy})$ in which $\bar{\Omega}^e_{xy}$ is the spatial domain of a typical element 'e' of the discretization. In computations of numerical solutions we choose

$$\hat{H} = L_0 = 3.175 \text{ mm} \ ; \ \rho_0 = \hat{\rho} = 800 \text{ kg/m}^3 \ ; \ \eta_0 = \hat{\eta} = 1.426 \text{ Pa s} \ ; \ u_0 = 0.5 \text{ m/s}$$

where $H = 1$, $P_0 = 200$ Pa, $Re = 0.8906$ and $De = 9.45$, same as model problem 1.

In this case, the rectangular domain $ABCD$ is discretized using two 9-node p-version elements of lengths 0.2 and 0.8 (figure 5.9) in the y-direction. Length AD is chosen as 1.0 (arbitrary). The local approximations are considered to be of equal degree for all variables. We consider $p = (p_1, p_2) = (9, 9)$ with $k = (k_1, k_2) = (2, 2)$, i.e. local approximations of class $C^{1,1}(\bar{\Omega}^e_{xy})$. For this choice of mesh, p-level and order of space, the residual functional values are of orders of $O(10^{-8})$ - $O(10^{-16})$ indicating that the PDEs are satisfied very accurately (in the pointwise sense for UCG1 as the integrals are Riemann, and not strictly in the pointwise sense for UCG2 since integrals are Lebesgue) when local approximations for u, v, p, τ_{xx}, τ_{yy} and τ_{xy} are of class $C^{1,1}(\bar{\Omega}^e_{xy})$.

In the numerical studies we begin with $\partial p/\partial x = -0.1$ for which a converged solution is ob-

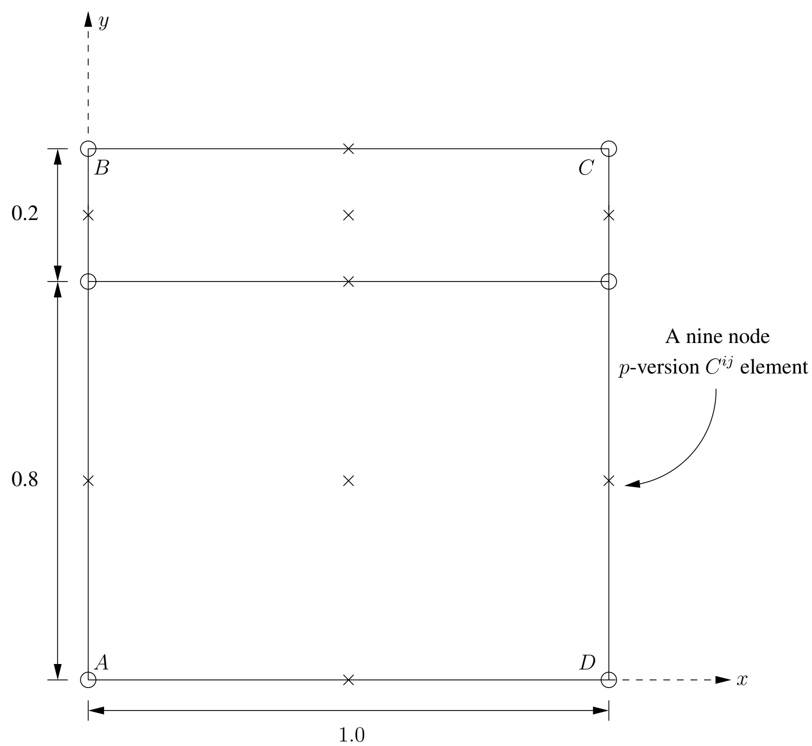


Figure 5.9: Graded mesh discretization using two 9-node p-version elements

tained and then progressively increase it up to $\partial p/\partial x = -0.275$ using a continuation procedure in which converged solutions at lower $\partial p/\partial x$ are used as initial (or starting) solution in the Newton's linear method. For all values of $\partial p/\partial x$, the computed numerical solutions confirm that u, $\partial u/\partial x$, τ_{xx}, τ_{yy} and τ_{xy} versus y are invariant of spatial location x along AD and are in perfect agreement with those obtained in model problem 1 using fully developed flow 1D numerical studies, hence are not repeated for sake of brevity.

5.9 Summary

We have presented development of rate constitutive theories for compressible and incompressible ordered thermoviscoelastic fluids in contravariant and covariant bases as well as using Jaumann rates. The theories consider convected time derivatives of up to order 'm' of the deviatoric Cauchy stress tensor and convected time derivatives of up to order 'n' of the strain tensor in the chosen basis. The convected time derivative of order 'm' of the deviatoric Cauchy stress tensor, the heat vector and Helmholtz free energy density are considered as dependent variables in the development of the rate constitutive theories. The argument tensors in the constitutive theories for the deviatoric stress tensor and heat vector are considered to be $[\gamma^{(j)}]$; $j = 1, 2, \dots, n$, $[{}_d\bar{\sigma}^{(k)}]$; $k = 0, 1, \dots, m-1$, density $\bar{\rho}$, temperature $\bar{\theta}$ and temperature gradient $\bar{\boldsymbol{g}}$ in the contravariant basis. In the case of covariant basis, $[\gamma^{(j)}]$ and $[{}_d\bar{\sigma}^{(k)}]$ are replaced by $[\gamma_{(j)}]$ and $[{}_d\bar{\sigma}_{(k)}]$ while the other arguments remain the same. When using Jaumann rates, we use $[{}^{(j)}\gamma^J]$; $j = 1, 2, \dots, n$ and $[{}^{(k)}_{\ d}\bar{\sigma}^J]$; $k = 0, 1, \dots, m-1$, $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$ as argument tensors. These rate constitutive theories define ordered thermoviscoelastic fluids of orders (m, n).

Many remarks made in papers by Surana et al. [55–57] regarding the second law of thermodynamics, conditions resulting from it, decomposition of the total stress tensor in equilibrium and deviatoric stress tensors, rates of stress and strain tensors in various bases, determination of equilibrium stress for incompressible and compressible cases leading to mechanical and thermodynamic pressure remain the same here as well and hence are not repeated for the sake of brevity. As in references [55–57], here also, the second law of thermodynamics does not provide a mechanism for determining the constitutive equations for the deviatoric stress tensor but only requires that the work expanded due to the deviatoric stress tensor be positive. The development of the rate constitutive theory presented in this chapter is based on the theory of generators and invariants. In this approach $[{}_d\bar{\sigma}^{(m)}]$ and $\bar{\boldsymbol{q}}^{(0)}$ or $[{}_d\bar{\sigma}_{(m)}]$ and $\bar{\boldsymbol{q}}_{(0)}$ or $[{}^{(m)}_{\ d}\bar{\sigma}^J]$ and ${}^{(0)}\bar{\boldsymbol{q}}^J$ are expressed as a linear combination of the combined generators of the argument tensors keeping in mind that $[{}_d\bar{\sigma}^{(m)}]$, $[{}_d\bar{\sigma}_{(m)}]$ and $[{}^{(m)}_{\ d}\bar{\sigma}^J]$ are symmetric tensors of rank two where as $\bar{\boldsymbol{q}}^{(0)}$, $\bar{\boldsymbol{q}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{q}}^J$ are tensors of rank one. Hence, the combined generators used in the linear combinations for $[{}_d\bar{\sigma}^{(m)}]$, $[{}_d\bar{\sigma}_{(m)}]$ or $[{}^{(m)}_{\ d}\bar{\sigma}^J]$ must also be symmetric tensors of rank two. Whereas the combined generators used to define $\bar{\boldsymbol{q}}^{(0)}$, $\bar{\boldsymbol{q}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{q}}^J$ must be tensors of rank one. Additionally we must also adhere to minimal basis in these linear combinations. The coefficients in the linear combinations are functions of density $\bar{\rho}$, temperature $\bar{\theta}$ and the combined invariants of the argument tensors of rank one and two and are determined by considering their Taylor series expansion about the configuration at time $t = t_n$ in the combined invariants and $\bar{\theta}$. We make the following specific remarks based on the work presented in this chapter.

1. The general theory of the rate constitutive equations for ordered thermoviscoelastic fluids of orders (m, n) is presented for compressible as well as incompressible thermoviscoelastic fluids.

2. The general theory of rate constitutive equations is specialized for $m = 1$ and $n = 1$, i.e., thermoviscoelastic fluids of order one in deviatoric Cauchy stress and strain rates. In this case $[{}_d\bar{\sigma}^{(1)}]$ or $[{}_d\bar{\sigma}_{(1)}]$ or $[{}^{(1)}_{d}\bar{\sigma}^J]$ contain $[{}_d\bar{\sigma}^{(0)}]$, $[\gamma^{(1)}]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ or $[{}_d\bar{\sigma}_{(0)}]$, $[\gamma_{(1)}]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ or $[{}^{(0)}_{d}\bar{\sigma}^J]$, $[{}^{(1)}\gamma^J]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ as argument tensors in contravariant and covariant bases as well as using Jaumann rates. The same argument tensors also hold for the heat vector $\bar{\boldsymbol{q}}^{(0)}$ or $\bar{\boldsymbol{q}}_{(0)}$ or ${}^{(0)}\bar{\boldsymbol{q}}^J$.

3. The general theory is also specialized for $m = 1$ and $n = 2$, i.e., thermoviscoelastic fluids of order one in deviatoric Cauchy stress rate but of order two in strain rate. In this case, the dependent variables $[{}_d\bar{\sigma}^{(1)}]$ or $[{}_d\bar{\sigma}_{(1)}]$ or $[{}^{(1)}_{d}\bar{\sigma}^J]$ contain $[{}_d\bar{\sigma}^{(0)}]$, $[\gamma^{(1)}]$, $[\gamma^{(2)}]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ or $[{}_d\bar{\sigma}_{(0)}]$, $[\gamma_{(1)}]$, $[\gamma_{(2)}]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ or $[{}^{(0)}_{d}\bar{\sigma}^J]$, $[{}^{(1)}\gamma^J]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ as argument tensors in contravariant, covariant and Jaumann bases. The same argument tensors also hold for the heat vector $\bar{\boldsymbol{q}}^{(0)}$ or $\bar{\boldsymbol{q}}_{(0)}$ or ${}^{(0)}\bar{\boldsymbol{q}}^J$.

4. The contravariant basis yields upper convected ordered rate constitutive theories. Likewise, covariant basis yields lower convected ordered rate constitutive theories. Use of Jaumann rates yield Jaumann rate constitutive equations. Surana et al. [30] have shown that only contravariant basis is in accordance with the physics of deforming matter when the deformation is finite. As the deformation deviates from the infinitesimal assumption, the rate constitutive equations based on covariant basis and others (such as Jaumann rate equations) become progressively spurious with progressively increasing deformation.

5. It is shown that Maxwell constitutive model and Giesekus constitutive model are a subset of ordered thermoviscoelastic fluids (incompressible) of orders $m = 1$ and $n = 1$. Derivations presented in the chapter demonstrate many assumptions needed for the general case of $m = 1$, $n = 1$ to derive these constitutive models. As well known, Maxwell model is a linear viscoelastic model whereas Giesekus constitutive model is a non-linear viscoelastic constitutive model.

6. It is also shown that Oldroyd-B constitutive model is a subset of the rate constitutive equations of orders $m = 1$ and $n = 2$. The derivation presented in this chapter demonstrates many assumptions that must be employed for the general case of $m = 1$ and $n = 2$ to derive Oldroyd-B constitutive model. This constitutive model is referred to as quasi-linear viscoelastic constitutive model. The non-linearity in this model is due to $[^{(2)}\gamma]$ (not considering the nonlinearity due to the variable coefficients).

7. The Maxwell, Oldroyd-B and Giesekus constitutive models as used in polymer science have been derived using kinetic theory [22, 51]. The reference to the Maxwell model based on continuum mechanics can be found in [3, 4]. However, the derivations of Oldroyd-B and Giesekus constitutive models based on principles and axioms of continuum mechanics as presented in this chapter are the first appearance of this work in the published literature to our knowledge.

8. The derivations of Maxwell, Oldroyd-B and Giesekus constitutive models presented here are fundamental in understanding the assumptions employed in their derivations which eventually limit their range of applications. For example, all three constitutive models might have limited range of validity for non-finite deformation. Giesekus model is superior in terms of more realistic behavior of $[{}_d\bar{\sigma}^{(0)}]$ or $[{}_d\bar{\sigma}_{(0)}]$ or $[{}^{(0)}_{d}\bar{\sigma}^J]$ due to inclusion of $[{}_d\bar{\sigma}^{(0)}]^2$ or $[{}_d\bar{\sigma}_{(0)}]^2$ or $[{}^{(0)}_{d}\bar{\sigma}^J]^2$ in the rate theory.

9. In this chapter we have also presented constitutive theories for $\bar{\boldsymbol{q}}^{(0)}$, $\bar{\boldsymbol{q}}_{(0)}$ and ${}^{(0)}\bar{\boldsymbol{q}}^J$ that contains same argument tensors as $[{}_d\bar{\sigma}^{(m)}]$, $[{}_d\bar{\sigma}_{(m)}]$ or $[{}^{(m)}_{d}\bar{\sigma}^J]$. This is essential for consistency

of the constitutive theories between the stress tensor and heat vector.

10. All developments consider the fluid to be compressible as well as incompressible in both contra- and co-variant bases as well as using Jaumann rates.

11. It is important to note that the Giesekus constitutive model derived here uses the deviatoric Cauchy stress tensor in the development of the rate theory. This is supported by the entropy inequality. The currently used Giesekus constitutive model in published works [22], though similar in the form compared to the model derived here, it uses deviatoric polymer Cauchy stress tensor in the constitutive model with the additional assumptions of (i) decomposition of deviatoric Cauchy stress tensor in solvent and polymer stress tensors (ii) Newton's law of viscosity to define the constitutive theory for the deviatoric solvent Cauchy stress tensor. These are not supported by the derivation presented in this chapter.

12. In polymer science, it is argued [69, 70] that decomposition of the deviatoric Cauchy stress in terms of viscous (both solvent and polymer) and elastic components and then expressing viscous stress using Newton's law of viscosity and thus obtaining constitutive equations in terms of deviatoric elastic stress is meritorious (computationally). This approach has two fundamental problems if viewed based on the principles and axioms of continuum mechanics for the constitutive theory. First, the deviatoric Cauchy stress must be a dependent variable in the constitutive theory and not the elastic stress as evident from entropy inequality. This argument questions the decomposition. Secondly, use of Newton's law of viscosity must be derivable as opposed to simply using it as a constitutive theory for the viscous stress tensor.

13. Numerical studies are presented for fully developed flow between parallel plates, and fully developed flow between parallel plates using a two dimensional formulation for a dense polymeric liquid (PIB/C14) using Giesekus constitutive model derived in this chapter as well as currently used Giesekus constitutive model. We use contravariant Cauchy stress tensor and Almansi strain tensors as conjugate measures of stress and strain in Eulerian description. This yields upper convected Giesekus constitutive models. Numerical results

are presented using the upper convected Giesekus constitutive model derived in this chapter (UCG1) as well as using the currently used constitutive model in deviatoric polymer stress (UCG2).

14. We choose a value of $\partial p/\partial x$ that is same in both constitutive models and compute results. For very low values of $\partial p/\partial x$, velocity fields in the two cases are not drastically different but as $\partial p/\partial x$ increases, both models produced significantly different velocity fields and hence different flow rates. Computed results from fully developed flow between parallel plates using studies in $\mathbb{R}^1$ and fully developed flow between parallel plates using two dimensional formulation i.e. $\mathbb{R}^2$ are in perfect agreement when the same constitutive model is used.

15. All rate theories presented here permit variable material coefficients during the deformation. Even though, the Maxwell, Oldroyd-B and Giesekus models can only be derived by neglecting many terms (as shown in the derivations) but the dependence of the final material coefficients can be maintained on any (or all) of the desired invariants. This feature permits shear thinning, shear thickening and other behaviors of viscosity etc. to be incorporated in the constitutive models derived here based on experimental and/or empirical relations.

16. A significant point to note in the present work is that determination of coefficients used in the linear combination of the generators to express deviatoric stress tensor or heat vector requires use of Taylor series expansion about the configuration at time $t = t_n$ when $t = t_{n+1}$ is the current configuration. This automatically forces the determination of the coefficients in the configuration at time $t = t_n$ and not in the current configuration at time $t = t_{n+1}$. In all presently used works, this is not the case. Variable transport properties as well as the dependence of the material coefficients on the invariants are all expressed using the current configuration. This may be justified when the configuration at $t = t_n$ and $t = t_{n+1}$ are in close proximity in terms of deformation field but cannot be supported by the derivation presented in this chapter.

Chapter 6

Summary and Conclusions

The rate constitutive theories in the Eulerian description for incompressible as well as compressible ordered thermoelastic solids, thermofluids and thermoviscoelastic fluids have been presented in contravariant and covariant bases as well as using the Jaumann rates. When the mathematical models for deforming matter are constructed using Eulerian description, the displacements of the material particles, and hence strain measures, are not readily obtainable. Thus the constitutive theories expressing chosen stress measures as a function of the conjugate strain measure is not usable. Hence, in this situation, one must consider a relationship between conjugate pairs of stress and strain rates, thus the need for rate constitutive theories.

Based on the axiom of admissibility, all constitutive equations must satisfy conservation laws to ensure thermodynamic equilibrium of the deforming matter. Since conservation of mass, balance of momenta and energy equation only require existence of the stress field and heat vector, these are independent of the constitution of the matter. Thus the second law of thermodynamics (Clausius-Duhem inequality) must provide the basis for the developments of the constitutive theories. The conditions resulting from the Clausius-Duhem inequality show that:

(i) Specific entropy is deterministic from the Helmholtz free energy and hence should not be considered as a dependent variable in the constitutive theory, thus the Cauchy stress tensor, heat vector and the Helmholtz free energy density are the only dependent variables in the constitutive theory for the type of matter considered here.

(ii) The conditions resulting from the entropy inequality also provide a mechanism to determine the heat vector as a function of the temperature gradient vector and conductivity, i.e. Fourier heat condition law.

(iii) However, these conditions do not provide a mechanism to determine the constitutive theory for the total Cauchy stress tensor. If the total Cauchy stress tensor is decomposed into equilibrium stress and deviatoric stress, then: (a) The equilibrium stress tensor is deterministic from the entropy inequality and leads to thermodynamic pressure for compressible matter and mechanical pressure for incompressible matter (b) The deviatoric Cauchy stress tensor is not deterministic from the entropy inequality, however the entropy inequality does require the work expanded due to the deviatoric Cauchy stress to be positive.

Thus, rate constitutive theories for ordered thermoelastic solids, thermofluids and thermoviscoelastic fluids reduce to the deviatoric Cauchy stress tensor, heat vector and the Helmholtz free energy density as dependent variables and determination of the theory for them in contra-, co-variant bases and the Jaumann rates using the argument tensors describing the physics of deformation.

For compressible ordered thermoelastic solids, the argument tensors of the first convected time derivative of the deviatoric Cauchy stress $[{}^{(1)}_{\ d}\bar{\sigma}]$ and heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$, the convected time derivatives of orders $1, 2, \dots, n$. These rate constitutive theories defined ordered thermoelastic solids of order n in the chosen basis.

In the case of compressible ordered thermofluids, the argument tensors of deviatoric Cauchy stress $[{}^{(0)}_{\ d}\bar{\sigma}]$ and heat vector ${}^{(0)}\bar{\boldsymbol{q}}$ are $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ and $[{}^{(j)}\gamma]$; $j = 1, 2, \dots, n$, defining ordered thermofluids of order n in contra- and covariant bases, and using Jaumann rates.

In the case of compressible thermoviscoelastic fluids, the argument tensors of the convected time derivative of order 'm' of the deviatoric Cauchy stress tensor and the heat vector are considered to be $[^{(j)}\gamma]$; $j = 1, 2, \dots, n$, $[^{(k)}_{d}\bar{\sigma}]$; $k = 0, 1, \dots, m-1$, $\bar{\rho}$, $\bar{\theta}$ and $\bar{\boldsymbol{g}}$. These rate constitutive theories define ordered thermoviscoelastic fluids of orders (m, n) in the chosen basis.

It is shown that the argument tensors for the Helmholtz free energy density $\bar{\Phi}$ are $\bar{\rho}$ and $\bar{\theta}$ regardless of the choice of basis and the type of matter and that for incompressible matter, $\bar{\rho}$ in the current configuration is same as in the reference configuration and hence it is no longer an argument of the dependent variables in the constitutive theories. Other arguments remain same as for the compressible case.

The theory of generators and invariants is utilized to derive the general form of the constitutive theories (both compressible and incompressible cases) in contravariant and covariant bases as well as using Jaumann rates. In this approach, the dependent variables (deviatoric Cauchy stress tensor or its convected time derivatives, and heat vector) are expressed as a linear combination of the combined generators of their argument tensors. We keep in mind that the deviatoric Cauchy stress tensor (and its convected time derivatives) in all three bases are symmetric tensors of rank two, hence the combined generators must also be symmetric tensors of rank two, whereas the heat vector is a tensor of rank one, thus the combined generators used to define the heat vector must be tensors of rank one. Additionally we must also adhere to integrity or minimal basis in these linear combinations. The coefficients in this linear combination are functions of the combined invariants of the argument tensors of rank one and two, in addition to $\bar{\rho}$ and $\bar{\theta}$ (in case of compressible matter) and $\bar{\theta}$ (in case of incompressible matter) in the current configuration at time $t = t_{n+1}$ and are determined by using their Taylor series expansions about a known configuration at time $t = t_n$ and retaining only up to linear terms in the combined invariants and the temperature.

General forms of rate constitutive theories for thermoelastic solids, thermofluids and thermoelastic solids are presented. The general form of the rate constitutive theory for thermoelastic solids of order one ($n = 1$) is specialized and derivations are presented for *generalized hypo-thermoelastic solids* and *hypo-thermoelastic solids* (both subsets of the rate constitutive theory of order one). It is shown that the constitutive theory for the deviatoric Cauchy stress tensor for *generalized Newtonian fluids* and *Newtonian fluids* is derivable using rate constitutive theory for thermofluids of order one ($n = 1$) with further assumptions and simplifications.

Specific details were presented for rate constitutive theories of orders $m = 1$ and $n = 1$, i.e. thermoviscoelastic fluids of order one in deviatoric Cauchy stress and strain rates. In this case $[{}^{(1)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ contain $[{}^{(0)}_{d}\bar{\sigma}]$, $[^{(1)}\gamma]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ as argument tensors in contravariant and covariant bases as well as using Jaumann rates. It was shown that *Maxwell constitutive model* and *Giesekus constitutive model* are a subset of ordered thermoviscoelastic fluids of orders $m = 1$ and $n = 1$. The Maxwell model is a linear viscoelastic model whereas Giesekus constitutive model is a non-linear viscoelastic constitutive model.

Specific details are also presented for rate constitutive theories of orders $m = 1$ and $n = 2$, i.e. thermoviscoelastic fluids of order one in deviatoric Cauchy stress rate but of order two in strain rate. In this case, the dependent variables $[{}^{(1)}_{d}\bar{\sigma}]$ and ${}^{(0)}\bar{\boldsymbol{q}}$ contain $[{}^{(0)}_{d}\bar{\sigma}]$, $[^{(1)}\gamma]$, $[^{(2)}\gamma]$, $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$ as argument tensors in contravariant, covariant and Jaumann bases. It was shown that *Oldroyd-B constitutive model* is a subset of the rate constitutive equations of orders $m = 1$ and $n = 2$. This constitutive model is referred to as quasi-linear viscoelastic constitutive model. The non-linearity in this model is due to $[^{(2)}\gamma]$ (not considering the nonlinearity due to the variable coefficients).

In this work, constitutive theories for the heat vector that contains same argument tensors as the deviatoric Cauchy stress tensor ($[{}^{(0)}_{d}\bar{\sigma}]$ for ordered thermofluids, $[{}^{(1)}_{d}\bar{\sigma}]$ in case of ordered thermoelastic solids, and $[{}^{(m)}_{d}\bar{\sigma}]$ for ordered thermoviscoelastic fluids) have been presented. This is essential for consistency of the constitutive theories between the stress tensor and heat vector.

All derivations are presented for compressible as well as incompressible case with variable material coefficients, which permit more complex and variable behavior of material coefficients during the evolution as they can be functions of density, temperature and the combined invariants of the argument tensors in the immediately preceding deformed configuration. Even though, the commonly used constitutive models can only be derived by neglecting many terms (as shown in the derivations) but the dependence of the final material coefficients or transport properties can be maintained on any (or all) of the desired invariants. This feature permits shear thinning, shear thickening and other behaviors of viscosity to be incorporated in the constitutive models derived here based on experimental or empirical relations and hence, it forms the basis for power law and Carreau-Yasuda models (and others) for shear rate dependent viscosity. In the following we draw some conclusions from the work presented here.

(1) Definition of $[{}_d\bar{\sigma}^{(1)}]$, $[{}_d\bar{\sigma}_{(1)}]$ and $[{}^{(1)}_{d}\bar{\sigma}^J]$ differ from each other. Furthermore, each of the three convected time derivatives have different definitions for compressible and incompressible cases. Thus, in the case of thermoelastic solids and thermoviscoelastic fluids of any orders, the resulting rate theories in contra- and co-variant bases and using the Jaumann rates would be different. The derivations of commonly used constitutive theories presented here for thermoelastic solids and thermoviscoelastic fluids are fundamental in understanding the assumptions employed in their derivations which eventually limit their range of applications:

   (a) When the first order rate constitutive equations ($n = 1$) for thermoelastic solids are simplified to obtain constitutive equations for what is commonly known as hypo thermoelastic solid, the restriction of infinitesimal deformation must be observed. In this case, second and higher order terms in the components of the first convected time derivatives of the strain tensor are assumed negligible. Thus, use of such constitutive relations [36, 37] for finite deformation may be questionable.

   (b) The assumptions used in deriving Maxwell, Oldroyd-B and Giesekus constitutive models are clearly stated. Giesekus model is superior in terms of more realistic behavior of $[{}^{(0)}_{\;d}\bar{\sigma}]$ due to inclusion of $[{}^{(0)}_{\;d}\bar{\sigma}]^2$ in the rate theory.

(2) For ordered thermofluids of order one ($n = 1$) i.e. when $[\gamma^{(1)}] = [\gamma_{(1)}] = [{}^{(1)}\gamma^J] = [\bar{D}]$ is the only argument tensor (in addition to $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$), the contra- and co-variant stress measures are the same, i.e. in this case $[{}_d\bar{\sigma}^{(0)}] = [{}_d\bar{\sigma}_{(0)}] = [{}^{(0)}_{\;d}\bar{\sigma}^J]$. For such fluids, the distinction between contra- and co-variant bases and Jaumann rates disappears and we may simply say deviatoric Cauchy stress $[{}_d\bar{\sigma}]$ as opposed to contra- and co-variant deviatoric Cauchy stress tensors or Jaumann stress tensor. Thus, for generalized Newtonian and Newtonian fluids (subsets of ordered thermofluids of order one) the covariant, contravariant and Jaumann stress measures are the same.

(3) The contravariant basis yields upper convected rate constitutive theories whereas covariant basis gives lower convected rate constitutive theories. Likewise, use of the Jaumann rates yields the Jaumann rate constitutive theories. Jaumann rate constitutive equations are most widely used for deforming solid matter in the Eulerian description [36, 37]. Surana et al. [30, 55, 56] have shown that the Jaumann stress rates are average of those in contra- and co-variant descriptions when the velocity field is the same in both bases and that in the case of finite deformation, only upper convected rate constitutive theory is in conformity with the physics of deformation [30]. As the magnitude of the deformation increases, the constitutive theories in covariant basis and others (such as Jaumann rate equations) become progressively more spurious as these use stress measures that do not correspond to the true deformed tetrahedron in the current configuration.

(4) A significant point to note in the present work is that determination of coefficients used in the linear combination of the generators to express the deviatoric stress tensor or heat vector requires use of Taylor series about the configuration at $t = t_n$ when $t = t_{n+1}$ is the current configuration. This automatically forces the determination of the coefficients in the configuration at time $t = t_n$ and not at $t = t_{n+1}$. In the majority of the published works,

this is not the case. In the published works, dependence of the variable transport properties i.e. material coefficients, on deformation field is expressed using the current configuration. This may be justified when the configurations at $t = t_n$ and $t = t_{n+1}$ are in close proximity in terms of deformation field but can not be supported by the derivations presented in this work.

(5) When using the theory of generators and invariants, the constitutive equation for the heat vector is much more complex due to the presence of other argument tensors than just $\bar{\boldsymbol{g}}$ when compared to Fourier heat conduction law which assumes that the heat vector only depends on $\bar{\boldsymbol{g}}$. When $[\bar{D}]$ is the only argument tensor of ${}^{(0)}\bar{\boldsymbol{q}}$ (in addition to $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{g}}$), then of course $\bar{\boldsymbol{q}}^{(0)} = \bar{\boldsymbol{q}}_{(0)} = {}^{(0)}\bar{\boldsymbol{q}}^J$ i.e. the constitutive theory for the heat vector is independent of the basis. However, the constitutive equation for the heat vector based on the combined generators of $[\bar{D}]$ and $\bar{\boldsymbol{g}}$ is perhaps more realistic for fluids and finite deformation of solids as it accounts for velocity gradients.

(6) The Giesekus constitutive model derived here uses the deviatoric Cauchy stress tensor in the development of the rate theory. This is supported by the entropy inequality. The currently used Giesekus constitutive model in published works [22], though similar in the form compared to the constitutive model derived here, it uses deviatoric polymer Cauchy stress tensor in the constitutive model with the additional assumptions of (i) decomposition of deviatoric Cauchy stress tensor in solvent and polymer stress tensors (ii) Newton's law of viscosity to define the constitutive theory for the deviatoric solvent Cauchy stress tensor. In polymer science, it is argued [69, 70] that this further decomposition of the deviatoric Cauchy stress is meritorious (computationally). This approach has two fundamental problems if viewed based on the principles and axioms of continuum mechanics for the constitutive theory. First, the deviatoric Cauchy stress must be a dependent variable in the constitutive theory and not the elastic stress as evident from entropy inequality. This argument questions the decomposition. Secondly, use of Newton's law of viscosity must be derivable as opposed to

simply using it as a constitutive theory for the viscous stress tensor.

(7) Numerical studies are presented using one dimensional studies for fully developed flow between parallel plates as well as two dimensional studies for a dense polymeric liquid (PIB/C14) using upper convected Giesekus constitutive model derived in this work (UCG1) as well as using the currently used upper convected constitutive model in deviatoric polymer stress (UCG2). Pressure gradient $\partial p/\partial x$ is chosen to be same in both constitutive models. For very low values of $\partial p/\partial x$, velocity fields in the two cases are not drastically different but as $\partial p/\partial x$ increases, both models produced significantly different velocity fields and hence different flow rates. Computed results from fully developed flow between parallel plates using studies in $\mathbb{R}^1$ and fully developed flow between parallel plates using two dimensional formulation i.e. in $\mathbb{R}^2$ are in perfect agreement when the same constitutive model is used.

(8) Numerical studies are also presented to determine the influence of non-linearity in terms of temperature gradient in the constitutive equation for the heat vector and comparisons are made with the Fourier heat conduction law. Heat transfer in an incompressible thermoelastic solid (aluminum rod) under small motion and infinitesimal deformation was considered. In this case, the distinction between covariant and contravariant basis disappears. The new coefficient associated with the non-linear term must be determined experimentally or empirically. The study demonstrates that when temperature gradients are high, the non-linear constitutive theory for the heat vector may be a more realistic representation of the physics as opposed to Fourier heat conduction law.

(9) Numerical studies are also presented for fully developed flow of a constant viscosity incompressible fluid (air) between parallel plates as model problem for a subset of thermofluids of order $n = 1$. The constitutive theory for the deviatoric Cauchy stress tensor that is quadratic in $[\bar{D}]$ is considered for numerical studies and compared with the constitutive theory based on Newton's law of viscosity. For a known pressure gradient, a theoretical solution is presented. The study suggests that when flow rates and hence velocity gradients are high, the

non-linear constitutive equations for the deviatoric Cauchy stress tensor may be a more realistic representation of the physics as opposed to the constitutive equations based on Newton's law of viscosity.

(10) All constitutive theories in Eulerian description in all three bases are in fact rate constitutive theories since they utilize convected time derivatives of the deviatoric Cauchy stress tensor and convected time derivatives of the strain tensor in the chosen basis in their derivations.

(11) The condition of positive work expanded due to deviatoric Cauchy stress tensor resulting from the entropy inequality must be satisfied by all rate constitutive equations. This work on constitutive inequalities is currently in progress.

(12) The constitutive theories in this work are derived using combined generators and invariants of the argument tensors of the dependent variables. Strictly speaking, these might be viewed to lack thermodynamic basis (as these are not derived using entropy inequality) unless supported by constitutive inequalities. However, these theories do have continuum mechanics foundation as they satisfy the axioms of the constitutive theory.

The work presented here provides completely general and unified constitutive theories for ordered thermoelastic solids, thermofluids and thermoviscoelastic fluids from which specialized behaviors such as (i) generalized hypo-thermoelastic solids and hypo-thermoelastic solids (ii) generalized Newtonian and Newtonian fluids (iii) Oldroyd-B, Maxwell and Giesekus constitutive models with variable material coefficients have been derived and presented.

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Appendix A

Combined generators and invariants

Table A.1: Complete set of irreducible invariants of symmetric tensor of rank two $[s]$, vector (tensor of rank one) $\{v\}$ and skew symmetric tensor $[w]$

(1) Invariants depending upon one variable:

Variables	**Invariants**
$[s]$	tr$[s]$, tr$[s^2]$, tr$[s^3]$
$\boldsymbol{v}$	$\boldsymbol{v} \cdot \boldsymbol{v}$
$[w]$	tr$[w^2]$

(2) Invariants depending upon two variables when (1) is assumed to hold:

Variables	**Invariants**
$[s_1]$, $[s_2]$	tr$([s_1][s_2])$, tr$([s_1^2][s_2])$, tr$([s_1][s_2^2])$
	tr$([s_1^2][s_2^2])$, tr$([s_1][s_2] + [s_2][s_1])$
	tr$([s_1][s_2] - [s_2][s_1])$
$[s]$, $\boldsymbol{v}$	$\boldsymbol{v} \cdot [s]\boldsymbol{v}$, $\boldsymbol{v} \cdot [s^2]\, \boldsymbol{v}$
$[s]$, $[w]$	tr$([s][w^2])$, tr$([s^2][w^2])$, tr$([s^2][w^2][s][w])$
$\boldsymbol{v}_1$, $\boldsymbol{v}_2$	$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$
$\boldsymbol{v}$, $[w]$	$\boldsymbol{v} \cdot [w^2]\boldsymbol{v}$
$[w_1]$, $[w_2]$	tr$([w_1][w_2])$

Table A.1: (Contd.) Complete set of irreducible invariants of symmetric tensor of rank two $[s]$, vector (tensor of rank one) $\{v\}$ and skew symmetric tensor $[w]$

(3) Invariants depending upon three variables when (1) and (2) are assumed to hold:

Variables	**Invariants**
$[s_1]\,,\quad [s_2]\,,\quad [s_3]$	$\mathrm{tr}([s_1][s_2][s_3])$
$[s_1]\,,\quad [s_2]\,,\quad \boldsymbol{v}$	$\boldsymbol{v}\cdot[s_1][s_2]\boldsymbol{v}$
$[s]\,,\quad \boldsymbol{v}_1\,,\quad \boldsymbol{v}_2$	$\boldsymbol{v}_1\cdot[s]\boldsymbol{v}_2\,,\quad \boldsymbol{v}_1\cdot[s^2]\boldsymbol{v}_2$
$[s]\,,\quad [w_1]\,,\quad [w_2]$	$\mathrm{tr}([s][w_1][w_2])\,,\quad \mathrm{tr}([s][w_1][w_2^2])\,,\quad \mathrm{tr}([s][w_1^2][w_2])$
$[s_1]\,,\quad [s_2]\,,\quad [w]$	$\mathrm{tr}([s_1][s_2][w])\,,\quad \mathrm{tr}([s_1^2][s_2][w])\,,$
	$\mathrm{tr}([s_1][w^2][s_2][w])\,,\quad \mathrm{tr}([s_1][s_2^2][w])$
$[w_1]\,,\quad [w_2]\,,\quad [w_3]$	$\mathrm{tr}([w_1][w_2][w_3])$
$\boldsymbol{v}_1\,,\quad \boldsymbol{v}_2\,,\quad [w]$	$\boldsymbol{v}_1\cdot[w]\boldsymbol{v}_2\,,\quad \boldsymbol{v}_1\cdot[w^2]\boldsymbol{v}_2$
$\boldsymbol{v}\,,\quad [w_1]\,,\quad [w_2]$	$\boldsymbol{v}\cdot[w_1][w_2]\boldsymbol{v}\,,\quad \boldsymbol{v}\cdot[w_1^2][w_2]\boldsymbol{v}\,,\quad \boldsymbol{v}\cdot[w_1][w_2^2]\boldsymbol{v}$
$[s]\,,\quad \boldsymbol{v}\,,\quad [w]$	$\boldsymbol{v}\cdot[s][w]\boldsymbol{v}\,,\quad \boldsymbol{v}\cdot[s^2][w]\boldsymbol{v}\,,\quad \boldsymbol{v}\cdot[w][s][w^2]\boldsymbol{v}$

(4) Invariants depending upon four variables when (1) to (3) are assumed to hold:

Variables	**Invariants**
$[s_1]\,,\ [s_2]\,,\ \boldsymbol{v}_1\,,\ \boldsymbol{v}_2$	$\boldsymbol{v}_1\cdot([s_1][s_2]-[s_2][s_1])\boldsymbol{v}_2$
$[s]\,,\ \boldsymbol{v}_1\,,\ \boldsymbol{v}_2\,,\ [w]$	$\boldsymbol{v}_1\cdot([s][w]-[w][s])\boldsymbol{v}_2$
$\boldsymbol{v}_1\,,\ \boldsymbol{v}_2\,,\ [w_1]\,,\ [w_2]$	$\boldsymbol{v}_1\cdot([w_1][w_2]-[w_2][w_1])\boldsymbol{v}_2$

Table A.2: Generators of rank one (vector valued isotropic functions)

(1) Generators depending upon one variable:

Variables	Generators
$\boldsymbol{v}$	$\boldsymbol{v}$

(2) Generators depending upon two variables when (1) is assumed to hold:

Variables	Generators
$[s]\,, \quad \boldsymbol{v}$	$[s]\boldsymbol{v}\,, \quad [s^2]\boldsymbol{v}$
$[w]\,, \quad \boldsymbol{v}$	$[w]\boldsymbol{v}\,, \quad [w^2]\boldsymbol{v}$

(3) Generators depending upon three variables when (1) and (2) are assumed to hold:

Variables	Generators
$[s_1]\,, \quad [s_2]\,, \quad \boldsymbol{v}$	$([s_1][s_2] + [s_2][s_1])\boldsymbol{v}\,, \quad ([s_1][s_2] - [s_2][s_1])\boldsymbol{v}$
$[w_1]\,, \quad [w_2]\,, \quad \boldsymbol{v}$	$([w_1][w_2] - [w_2][w_1])\boldsymbol{v}$
$[s]\,, \quad \boldsymbol{v}\,, \quad [w]$	$([s][w] - [w][s])\boldsymbol{v}$

Table A.3: Generators for symmetric tensor-valued isotropic functions

(1) Generators depending upon no variables: $[I]$

(2) Generators depending upon one variable:

Variables	Generators
$[s]$	$[s]\ ,\quad [s^2]$
$\boldsymbol{v}$	$\boldsymbol{v}\otimes\boldsymbol{v}$
$[w]$	$[w^2]$

(3) Generators depending upon two variables when (1) is assumed to hold:

Variables	Generators
$[s_1]\ ,\quad [s_2]$	$[s_1][s_2]+[s_2][s_1]\ ,\quad [s_1^2][s_2]+[s_1][s_2^2]\ ,$
	$[s_1][s_2^2]+[s_2^2][s_1]$
$[s]\ ,\quad \boldsymbol{v}$	$\boldsymbol{v}\otimes[s]\boldsymbol{v}+[s]\boldsymbol{v}\otimes\boldsymbol{v}\ ,\quad \boldsymbol{v}\otimes[s^2]\boldsymbol{v}+[s^2]\boldsymbol{v}\otimes\boldsymbol{v}$
$[s]\ ,\quad [w]$	$[s][w]-[w][s]\ ,\quad [w][s][w]\ ,$
	$[s^2][w]-[w][s^2]\ ,\quad [w][s][w^2]-[w^2][s][w]$
$\boldsymbol{v}_1\ ,\quad \boldsymbol{v}_2$	$\boldsymbol{v}_1\otimes\boldsymbol{v}_2+\boldsymbol{v}_2\otimes\boldsymbol{v}_1$
$\boldsymbol{v}\ ,\quad [w]$	$[w]\boldsymbol{v}\otimes[w]\boldsymbol{v}\ ,\quad \boldsymbol{v}\otimes[w]\boldsymbol{v}+[w]\boldsymbol{v}\otimes\boldsymbol{v}\ ,$
	$[w]\boldsymbol{v}\otimes[w^2]\boldsymbol{v}+[w^2]\boldsymbol{v}\otimes[w]\boldsymbol{v}$
$[w_1]\ ,\quad [w_2]$	$[w_1][w_2]+[w_2][w_1]\ ,\quad [w_1][w_2^2]-[w_2^2][w_1]\ ,$
	$[w_1^2][w_2]-[w_2][w_1^2]$

(4) Generators depending upon three variables when (1) and (2) are assumed to hold:

Variables	Generators
$[s]\ ,\quad \boldsymbol{v}_1\ ,\quad \boldsymbol{v}_2$	$[s](\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)-(\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)[s]$
$[w]\ ,\quad \boldsymbol{v}_1\ ,\quad \boldsymbol{v}_2$	$[w](\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)+(\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)[w]$

Table A.4: Generators for skew symmetric tensor-valued isotropic functions

(1) Generators depending upon one variable:

Variables	**Generators**
$[w]$	$[w]$

(2) Generators depending upon two variables when (1) is assumed to hold:

Variables	**Generators**
$[s_1]\ ,\quad [s_2]$	$[s_1][s_2]-[s_2][s_1]\ ,\quad [s_1^2][s_2]-[s_2][s_1^2]\ ,$
	$[s_1][s_2^2]-[s_2^2][s_1]\ ,\quad [s_1][s_2][s_1^2]-[s_1^2][s_2][s_1]\ ,$
	$[s_2][s_1][s_2^2]-[s_2^2][s_1][s_2]$
$[s]\ ,\quad \boldsymbol{v}$	$\boldsymbol{v}\otimes[s]\boldsymbol{v}-[s]\boldsymbol{v}\otimes\boldsymbol{v}\ ,\quad \boldsymbol{v}\otimes[s^2]\boldsymbol{v}-[s^2]\boldsymbol{v}\otimes\boldsymbol{v}$
	$[s]\boldsymbol{v}\otimes[s^2]\boldsymbol{v}-[s^2]\boldsymbol{v}\otimes[s]\boldsymbol{v}$
$[s]\ ,\quad [w]$	$[s][w]+[w][s]\ ,\quad [s][w^2]-[w^2][s]$
$[w]\ ,\quad \boldsymbol{v}$	$\boldsymbol{v}\otimes[w]\boldsymbol{v}-[w]\boldsymbol{v}\otimes\boldsymbol{v}\ ,\quad \boldsymbol{v}\otimes[w^2]\boldsymbol{v}-[w^2]\boldsymbol{v}\otimes\boldsymbol{v}$
$\boldsymbol{v}_1\ ,\quad \boldsymbol{v}_2$	$\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1$
$[w_1]\ ,\quad [w_2]$	$[w_1][w_2]-[w_2][w_1]$

(3) Generators depending upon three variables when (1) and (2) are assumed to hold:

Variables	**Generators**
$[s_1]\ ,\quad [s_2]\ ,\quad [s_3]$	$[s_1][s_2][s_3]+[s_2][s_3][s_1]+[s_3][s_1][s_2]-$
	$[s_3][s_2][s_1]-[s_1][s_3][s_2]-[s_2][s_1][s_3]$
$[s_1]\ ,\quad [s_2]\ ,\quad \boldsymbol{v}$	$[s_1]\boldsymbol{v}\otimes[s_2]\boldsymbol{v}-[s_2]\boldsymbol{v}\otimes[s_1]\boldsymbol{v}+$
	$\boldsymbol{v}\otimes([s_1][s_2]-[s_2][s_1])\boldsymbol{v}-([s_1][s_2]-[s_2][s_1])\boldsymbol{v}\otimes\boldsymbol{v}$
$[s]\ ,\quad \boldsymbol{v}_1\ ,\quad \boldsymbol{v}_2$	$[s](\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)+(\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)[s]$
$[w]\ ,\quad \boldsymbol{v}_1\ ,\quad \boldsymbol{v}_2$	$[w](\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)-(\boldsymbol{v}_1\otimes\boldsymbol{v}_2-\boldsymbol{v}_2\otimes\boldsymbol{v}_1)[w]$